Dynamics Mixed Exercise 5

1 a $F = ma$

$-\dfrac{k}{(x+2)^2} = 0.6a$

The force is a function of x so use $a = v\dfrac{dv}{dx}$.

$0.6v\dfrac{dv}{dx} = -\dfrac{k}{(x+2)^2}$

$0.6\int v\ dv = -\int \dfrac{k}{(x+2)^2}dx$

Separate the variables and integrate.

$0.3v^2 = \dfrac{k}{(x+2)} + c$

$x = 2, v = 8 \quad 0.3 \times 8^2 = \dfrac{k}{4} + c$

$x = 10, v = 2 \quad 0.3 \times 2^2 = \dfrac{k}{12} + c$

Use the given information to obtain a pair of simultaneous equations in k and c.

$\text{Subtract}: 0.3(8^2 - 2^2) = \dfrac{k}{4} - \dfrac{k}{12}$

$0.3 \times 60 = \dfrac{k}{6}$

Solve to find k.

$k = 0.3 \times 60 \times 6 = 108$

b From above $0.3 \times 4 = \dfrac{k}{12} + c$

$c = 1.2 - \dfrac{108}{12} = -7.8$

Find c to complete the expression for v^2.

$\therefore \quad 0.3v^2 = \dfrac{108}{(x+2)} - 7.8$

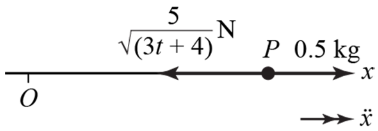
$v = 0 \quad 0 = \dfrac{108}{x+2} - 7.8$

$7.8(x+2) = 108$

$x = \dfrac{108}{7.8} - 2 = 11.84\ldots$

The distance OB is 11.8 m (3 s.f.).

2 a



$$F = ma$$

$$-\frac{5}{\sqrt{(3t+4)}} = 0.5\ddot{x}$$

$$\ddot{x} = -10(3t+4)^{-\frac{1}{2}}$$

$$\dot{x} = -\frac{10}{\frac{1}{2}\times 3}(3t+4)^{\frac{1}{2}} + c$$

$$t = 0 \quad \dot{x} = 12 \quad 12 = -\frac{20}{3}\sqrt{4} + c$$

$$c = 12 + \frac{40}{3} = \frac{76}{3}$$

$$\therefore \dot{x} = -\frac{20}{3}(3t+4)^{\frac{1}{2}} + \frac{76}{3}$$

b $x = -\frac{20}{3\times\frac{3}{2}\times 3}(3t+4)^{\frac{3}{2}} + \frac{76}{3}t + A$ ← Integrate line above.

$$t = x = 0 \therefore A = \frac{40}{27}\times 4^{\frac{3}{2}} = \frac{320}{27}$$

P at rest $\Rightarrow \frac{76}{3} = \frac{20}{3}(3t+4)^{\frac{1}{2}}$ ← Using result from **a**.

$$\left(\frac{76}{20}\right)^2 = 3t+4$$

$$t = \frac{1}{3}\left[\left(\frac{76}{20}\right)^2 - 4\right]$$

$$t = 3.48$$

When $t = 3.48$

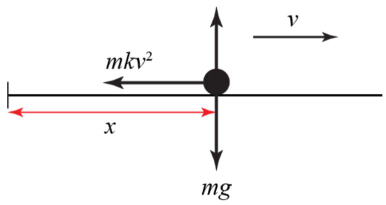
$$x = -\frac{40}{27}\left(\frac{76}{20}\right)^3 + \frac{76}{3}\times 3.48 + \frac{320}{27}$$

← $3t+4 = \left(\frac{76}{20}\right)^2$, so use the exact value here.

$$x = 18.72$$

P is 18.7 m from O (3 s.f.)

3



$R(\rightarrow)\,\mathbf{F} = m\mathbf{a}$

$-mkv^2 = ma$

$-kv^2 = \frac{dv}{dt}$

Separating the variables

$\int k\,dt = -\int v^{-2}\,dv$

$kt = -\frac{v^{-1}}{-1} + A = \frac{1}{v} + A$

At $t = 0,\ v = U$

$0 = \frac{1}{U} + A \Rightarrow A = -\frac{1}{U}$

Hence

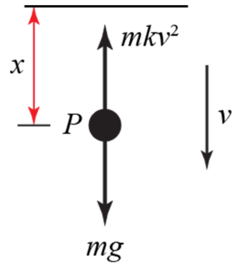
$t = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{U}\right)$

When $v = \frac{3}{4}U$

$t = \frac{1}{k}\left(\frac{1}{\frac{3}{4}U} - \frac{1}{U}\right) = \frac{1}{k}\left(\frac{4}{3U} - \frac{1}{U}\right) = \frac{1}{3kU}$

The time at which the particle's speed is $\frac{3}{4}U$ is $\frac{1}{3kU}$.

4



$R(\downarrow)\mathbf{F} = m\mathbf{a}$

$$mg - mkv^2 = ma$$

$$g - kv^2 = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x = \int \frac{v}{g - kv^2}\,\mathrm{d}v$$

$$x = A - \frac{1}{2k}\ln(g - kv^2)$$

When $x = 0,\ v = 0$

$$0 = A - \frac{1}{2k}\ln g \Rightarrow A = \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln(g - kv^2) = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$\ln\left(\frac{g}{g - kv^2}\right) = 2kx$$

$$\frac{g}{g - kv^2} = \mathrm{e}^{2kx}$$

$$g - kv^2 = g\,\mathrm{e}^{-2kx}$$

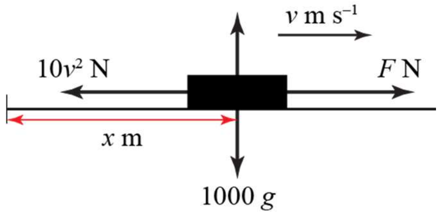
$$v^2 = \frac{g}{k}(1 - \mathrm{e}^{-2kx})$$

When $x = D$

$$v^2 = \frac{g}{k}(1 - \mathrm{e}^{-2kD})$$

$$v = \left(\frac{g}{k}\right)^{\frac{1}{2}}(1 - \mathrm{e}^{-2kD})^{\frac{1}{2}}$$

5



$12\text{ kW} = 12\ 000\text{W}$

$$P = Fv$$

$$F = \frac{12\,000}{v}$$

$$\text{R}(\rightarrow)\mathbf{F} = m\mathbf{a}$$

$$F - 10v^2 = 1000a$$

$$\frac{12\,000}{v} - 10v^2 = 1000v\frac{\mathrm{d}y}{\mathrm{d}x}$$

Dividing throughout by 10 and multiplying throughout by v

$$1200 - v^3 = 100v^2\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x = 100\int \frac{v^2}{1200 - v^3}\,\mathrm{d}v$$

$$x = A - \frac{100}{3}\ln(1200 - v^3)$$

Let $x = 0$ when $v = 5$

$$0 = A - \frac{100}{3}\ln(1200 - 125) \Rightarrow A = \frac{100}{3}\ln 1075$$

Hence

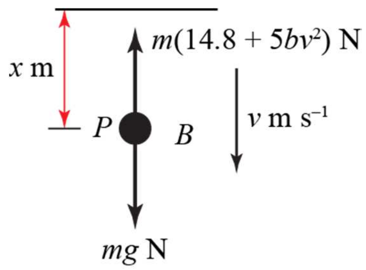
$$x = \frac{100}{3}\ln 1075 - \frac{100}{3}\ln(1200 - v^3) = \frac{100}{3}\ln\left(\frac{1075}{1200 - v^3}\right)$$

When $v = 10$

$$x = \frac{100}{3}\ln\left(\frac{1075}{1200 - 10^3}\right) = \frac{100}{3}\ln\left(\frac{1075}{200}\right) \approx 56.1$$

The distance travelled as the car's speed increases from 5 m s^{-1} to 10 m s^{-1} is 56.1 m (3 s.f.).

6



R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - m(14.8 + 5bv^2) = ma$$

$$9.8 - 14.8 - 5bv^2 = v\frac{dv}{dx}$$

$$-5(1 + bv^2) = v\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = -\frac{1}{5}\int \frac{v}{1 + bv^2}\,dv$$

$$x = A - \frac{1}{10b}\ln(1 + bv^2)$$

At $x = 0$, $v = U$

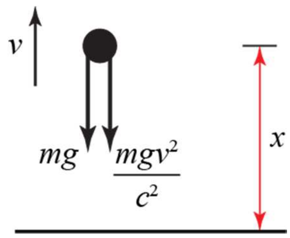
$$0 = A - \frac{1}{10b}\ln(1 + bU^2) \Rightarrow A = \frac{1}{10b}\ln(1 + bU^2)$$

$$x = \frac{1}{10b}\ln(1 + bU^2) - \frac{1}{10b}\ln(1 + bv^2) = \frac{1}{10b}\ln\left(\frac{1 + bU^2}{1 + bv^2}\right)$$

When $v = 0$, $x = d$

$$d = \frac{1}{10b}\ln(1 + bU^2)$$

7 a



R(↑) $\quad \mathbf{F} = m\mathbf{a}$

$$-mg - \frac{mgv^2}{c^2} = ma$$

$$-g\left(\frac{c^2+v^2}{c^2}\right) = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\int g\,\mathrm{d}x = -c^2\int \frac{v}{c^2+v^2}\,\mathrm{d}v$$

$$gx = A - \frac{c^2}{2}\ln(c^2+v^2)$$

At $x = 0,\ v = V$

$$0 = A - \frac{c^2}{2}\ln(c^2+V^2) \Rightarrow A = \frac{c^2}{2}\ln(c^2+V^2)$$

Hence

$$gx = \frac{c^2}{2}\ln(c^2+V^2) - \frac{c^2}{2}\ln(c^2+v^2) = \frac{c^2}{2}\ln\left(\frac{c^2+V^2}{c^2+v^2}\right)$$

$$x = \frac{c^2}{2g}\ln\left(\frac{c^2+V^2}{x^2+v^2}\right)$$

At the greatest height $v = 0$

$$x = \frac{c^2}{2g}\ln\left(\frac{c^2+V^2}{c^2}\right) = \frac{c^2}{2g}\ln\left(1+\frac{V^2}{c^2}\right),$$ as required.

7 b $R(\uparrow)$ $\quad \mathbf{F} = ma$

$$-mg - \frac{mgv^2}{c^2} = ma$$

$$-g\left(\frac{c^2+v^2}{c^2}\right) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variable

$$\frac{g}{c^2}\int 1\,\mathrm{d}t = -\int \frac{1}{c^2+v^2}\,\mathrm{d}v$$

$$\frac{gt}{c^2} = A - \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

When $t = 0, v = V$

$$0 = A - \frac{1}{c}\arctan\left(\frac{V}{c}\right) \Rightarrow A = \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

Hence

$$\frac{gt}{c^2} = \frac{1}{c}\arctan\left(\frac{V}{c}\right) - \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

At the greatest height $v = 0$

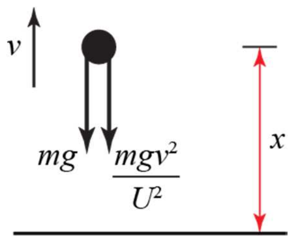
$$\frac{gt}{c^2} = \frac{1}{c}\arctan\left(\frac{V}{c}\right) \Rightarrow t = \frac{c}{g}\arctan\left(\frac{V}{c}\right)$$

The time taken to reach the greatest height is $\frac{c}{g}\arctan\left(\frac{V}{c}\right)$.

8 a Let the mass of the particle be m.
Let the resistance be kv^2, where k is a constant of proportionality.
If U is the speed for which the resistance is equal to the weight of the particle then

$$kU^2 = mg \Rightarrow k = \frac{mg}{U^2}$$

Hence the resistance is $\dfrac{mgv^2}{U^2}$.



R(↑) $\mathbf{F} = m\mathbf{a}$

$$-mg - \frac{mgv^2}{U^2} = ma$$

$$-\frac{g(U^2+v^2)}{U^2} = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \textbf{(1)}$$

Separating the variables

$$\int g \,\mathrm{d}t = -U^2 \int \frac{1}{U^2+v^2}\,\mathrm{d}v$$

$$gt = A - U^2 \times \frac{1}{U}\arctan\left(\frac{v}{U}\right)$$

When $t = 0$, $v = U$

$$0 = A - U\arctan 1 \Rightarrow A = U\arctan 1 = \frac{\pi U}{4}$$

Hence

$$gt = \frac{\pi U}{4} - U\arctan\left(\frac{v}{U}\right)$$

$$t = \frac{\pi U}{4g} - \frac{U}{g}\arctan\left(\frac{v}{U}\right)$$

Let the time of ascent be T.
When $t = T$, $v = 0$

$$T = \frac{\pi U}{4g} - \frac{U}{g}\arctan 0$$

$$= \frac{\pi U}{4g}, \text{ as required}$$

8 b Equation **(1)** in part **a** can be written as

$$-\frac{g(U^2+v^2)}{U^2}=v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

Equation *in part a can be written as

$$-\frac{g(U^2+v^2)}{U^2}=v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int g\,dx=-U^2\int\frac{v}{U^2+v^2}\mathrm{d}v$$

$$gx=B-\frac{U^2}{2}\ln(U^2+v^2)$$

When $x=0, v=U$

$$0=B-\frac{U^2}{2}\ln(2U^2)\Rightarrow B=\frac{U^2}{2}\ln(2U^2)$$

Hence

$$gx=\frac{Ux^2}{2}\ln(2U^2)-\frac{U^2}{2}\ln(U^2+v^2)$$

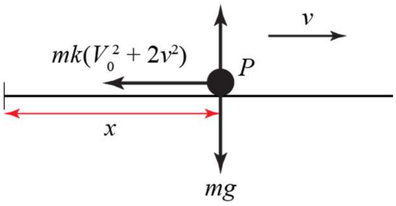
$$x=\frac{U^2}{2}\ln\left(\frac{2U^2}{U^2+v^2}\right)$$

Let the total distance ascended be H.

When $h=H, v=0$

$$H=\frac{U^2}{2g}\ln\left(\frac{2U^2}{U^2}\right)=\frac{U^2}{2g}\ln 2, \text{as required}$$

9 a



R(→) $\quad$ $\mathbf{F} = m\mathbf{a}$

$-mk(V_0^2 + 2v^2) = ma$

$-k(V_0^2 + 2v^2) = v\dfrac{dv}{dx}$ **(1)**

Separating the variables

$\int k\ dx = -\int \dfrac{v}{V_0^2 + 2v^2} dv$

$kx = A - \dfrac{1}{4}\ln(V_0^2 + 2v^2)$

At $x = 0,\ v = V_0$

$0 = A - \dfrac{1}{4}\ln(V_0^2 + 2V_0^2) \Rightarrow A = \dfrac{1}{4}\ln(3V_0^2)$

Hence

$kx = \dfrac{1}{4}\ln(3V_0^2) - \dfrac{1}{4}\ln(V_0^2 + 2v^2)$

$x = \dfrac{1}{4k}\ln\left(\dfrac{3V_0^2}{V_0^2 + 2v^2}\right)$

When $v = \dfrac{1}{2}V_0$

$x = \dfrac{1}{4k}\ln\left(\dfrac{3V_0^2}{V_0^2 + \frac{1}{2}V_0^2}\right) = \dfrac{1}{4k}\ln\left(\dfrac{3V_0^2}{\frac{3}{2}V_0^2}\right)$

$= \dfrac{\ln 2}{4k}$, as required.

9 b Equation **(1)** can be written as

$$-k(v_0^2 + 2v^2) = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\int k\,dt = -\int \frac{1}{V_0^2 + 2v^2}\,dv = -\frac{1}{2}\int \frac{1}{\left(\frac{V_0}{\sqrt{2}}\right)^2 + v^2}\,\mathrm{d}v$$

$$kt = B - \frac{1}{2} \times \frac{1}{\left(\frac{V_0}{\sqrt{2}}\right)} \arctan \frac{v}{\left(\frac{V_0}{\sqrt{2}}\right)}$$

When $t = 0, v = V_0$

$$0 = B - \frac{\sqrt{2}}{2V_0} \arctan\left(\frac{\sqrt{2}V_0}{V_0}\right) \Rightarrow B = \frac{\sqrt{2}}{2V_0} \arctan\sqrt{2}$$

Hence

$$t = \frac{\sqrt{2}}{2kV_0}\left(\arctan\sqrt{2} - \arctan\left(\frac{\sqrt{2}v}{V_0}\right)\right)$$

$$v = \frac{1}{2}V_0$$

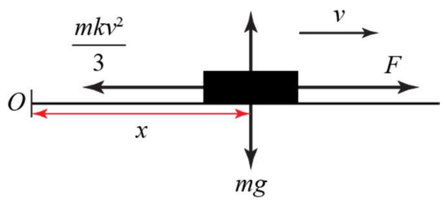
$$t = \frac{\sqrt{2}}{2kV_0}\left(\arctan\sqrt{2} - \arctan\left(\frac{\sqrt{2} \times \frac{1}{2}V_0}{V_0}\right)\right)$$

$$= \frac{1}{kV_0}\left[\frac{\sqrt{2}}{2}\left(\arctan\sqrt{2} - \arctan\left(\frac{\sqrt{2}}{2}\right)\right)\right]$$

This has the form $\frac{\lambda}{kV_0}$, as required, where

$$\lambda = \frac{\sqrt{2}}{2}\left(\arctan\sqrt{2} - \arctan\left(\frac{\sqrt{2}}{2}\right)\right) \approx 0.24 (2\,\text{d.p.})$$

10



a $P = Fv \Rightarrow F = \frac{P}{v}$

$R(\rightarrow)$ $\mathbf{F} = m\mathbf{a}$

$F - \frac{mkv^2}{3} = ma$

$\frac{P}{v} - \frac{mkv^2}{3} = mv\frac{dv}{dx}$

Multiplying throughout by $3v$

$3P - mkv^3 = 3mv^2\frac{dv}{dx}$

$3mv^2\frac{dv}{dx} = 3P - mkv^3$, as required

b The limiting speed is given by $a = v\frac{dv}{dx} = 0$

$0 = 3P - mkv^3 \Rightarrow v^3 = \frac{3P}{mk} \Rightarrow v = \left(\frac{3P}{mk}\right)^{\frac{1}{3}}$

Separating the variables in the answer to part **a**

$\int 1\, dx = \int \frac{3mv^2}{3P - mkv^3} dv$

$x = A - \frac{1}{k}\ln(3P - mkv^3)$

When $x = 0, v = \frac{1}{2}\left(\frac{3P}{mk}\right)^{\frac{1}{3}} \Rightarrow v^3 = \frac{3P}{8mk}$

$0 = A - \frac{1}{k}\ln\left(3P - \frac{3P}{8}\right) \Rightarrow A = \frac{1}{k}\ln\left(\frac{21P}{8}\right)$

Hence

$x = \frac{1}{k}\ln\left(\frac{21P}{8}\right) - \frac{1}{k}\ln(3P - mkv^3)$

$= \frac{1}{k}\ln\left(\frac{21P}{8(3P - mkv^3)}\right)$

11 a $F = \dfrac{k}{x^2}$

when $x = R$, $F = \text{mg}$

$\therefore \dfrac{k}{R^2} = mg,\ k = mg\,R^2$

When $x = R$, S is on the surface of the Earth and the force exerted by the Earth on S is mg.

b $\text{Force} = \dfrac{mg\,R^2}{x^2}$

$F = ma$

$-\dfrac{mgR^2}{x^2} = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$

$-\int \dfrac{gR^2}{x^2}\,\mathrm{d}x = \int v\,\mathrm{d}v$

$\dfrac{1}{2}v^2 = \dfrac{g\,R^2}{x} + c$

$x = 5R,\ v = 0\ c = \dfrac{-g\,R^2}{5R}$

$\therefore v^2 = 2g\dfrac{R^2}{x} - \dfrac{2g\,R^2}{5R}$

When $x = R$ $v^2 = 2g\dfrac{R^2}{R} - \dfrac{2g\,R^2}{5R}$

$v^2 = \dfrac{8Rg}{5}$

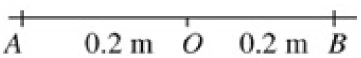
The speed of the spacecraft is

$\sqrt{\left(\dfrac{8Rg}{5}\right)} = 2\sqrt{\left(\dfrac{2Rg}{5}\right)}$

The force is in the direction of *decreasing* x.

The force is a function of x so use $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$.

12



a amplitude $= 0.4 \div 2 = 0.2\,\text{m}$

period $= 2 \times 2.5 = 5\text{s}$

$\therefore\ \dfrac{2\pi}{\omega} = 5\quad \omega = \dfrac{2\pi}{5}$

$v^2 = \omega^2(a^2 - x^2)$

$u^2 = \left(\dfrac{2\pi}{5}\right)^2(0.2^2 - 0)$

$u = \dfrac{2\pi}{5} \times 0.2 = \dfrac{4\pi}{50}(\text{or } 0.2513\ldots)$

$\therefore u = \dfrac{4\pi}{50}(\text{or } 0.251\ (3\text{s.f.}))$

Find a and ω from the given information.

Now use $v^2 = \omega^2(a^2 - x^2)$ to find u.

12 b $x = a\sin\omega t$

$x = 0.2\sin\left(\frac{2\pi}{5}t\right)$

P is at the centre of oscillation when $t = 0$.

$\dot{x} = \frac{2\pi}{5}\times 0.2\cos\left(\frac{2\pi}{5}t\right)$

Differentiate x with respect to t to find $\dot{x}$.

$t = 3$ $\dot{x} = \frac{0.4\pi}{5}\cos\frac{6\pi}{5} = -0.2033$

When $t = 3$ P's speed is 0.203 m^{-1} (3 s.f.).

Speed is positive.

c $x = a\sin\omega t$

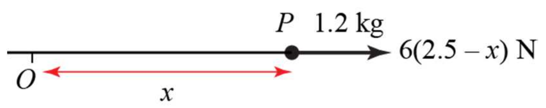
$x = 0.2\sin\left(\frac{2\pi}{5}t\right)$

$t = 3$ $x = 0.2\sin\left(\frac{6\pi}{5}\right) = -0.1175\ldots$

$\therefore$ Distance from A is $0.2 + 0.1175\ldots$
$= 0.318$ m (3 s.f.)

P is moving towards A when $t = 0$, so x is negative between O and B.

13



a $x = 2.5$

The acceleration (and therefore the resultant force) are zero when the speed is maximum.

b $F = ma$

$6(2.5 - x) = 1.2a$

$6(2.5 - x) = 1.2v\frac{\text{d}v}{\text{d}x}$

When the force is a function of x use $a = v\frac{\text{d}v}{\text{d}x}$.

$v\frac{\text{d}v}{\text{d}x} = 5(2.5 - x)$

$\int v\,\text{d}v = \int 5(2.5 - x)\,\text{d}x$

Separate the variables.

$\frac{1}{2}v^2 = 5\left(2.5x - \frac{x^2}{2}\right) + C$

Integrate. Don't forget the constant!

$x = 2.5$ $v = 8$

$\frac{1}{2}\times 8^2 = 5\left(2.5\times 2.5 - \frac{2.5^2}{2}\right) + C$

a gives the initial conditions.

$C = 32 - 5\times\frac{2.5^2}{2} = 16.375$

$v^2 = 10\left(2.5x - \frac{x^2}{2}\right) + 2\times 16.375$

$v^2 = 25x - 5x^2 + 32.75$

14 a $x = 3\sin\left(\frac{\pi}{4}t\right)$

$\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

$\ddot{x} = -3\left(\frac{\pi}{4}\right)^2 \sin\left(\frac{\pi}{4}t\right)$

Differentiate $x = 3\sin\left(\frac{\pi}{4}t\right)$ twice.

$\ddot{x} = -\left(\frac{\pi}{4}\right)^2 x$

Obtain the equation of the form $\ddot{x} = -\omega^2 x$.

$\therefore$ S.H.M.

b amplitude $= 3\,m$

$\text{period} = \frac{2\pi}{\omega} = 2\pi \times \frac{4}{\pi} = 8\text{s}$

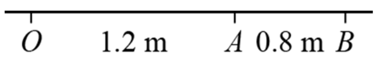
c From **a** $\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

Or use $v_{max} = a\omega$.

$\Rightarrow$ maximum speed $= \frac{3\pi}{4}\,\text{m s}^{-1}$

(or $2.36\ \text{m s}^{-1}$ (3s.f.))

d



$x = 3\sin\left(\frac{\pi}{4}t\right)$

At A, $x = 1.2 \quad 1.2 = 3\sin\left(\frac{\pi}{4}t_a\right)$

$t_a = \frac{4}{\pi}\sin^{-1}\left(\frac{1.2}{3}\right)$

At B, $x = 2\ \ t_b = \frac{4}{\pi}\sin^{-1}\left(\frac{2}{3}\right)$

$\text{Time } A \to B = \frac{4}{\pi}\left[\sin^{-1}\left(\frac{2}{3}\right) - \sin^{-1}\left(\frac{1.2}{3}\right)\right]$

$= 0.4051$

Leave calculator work as late as possible.

The time to go directly from A to B is 0.405 s (3 s.f.)

15 a $\ddot{x} = -\omega^2 x$

When $x = 0.09$, $\ddot{x} = -1.5$

$-1.5 = -\omega^2 \times 0.09$

$\omega^2 = \dfrac{50}{3}$

$\omega = \sqrt{\dfrac{50}{3}} = 4.08$

$T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{3}{50}} = 1.54 \text{ s}$

b $v^2 = \omega^2\left(a^2 - x^2\right)$

$0.3^2 = \dfrac{50}{3}\left(a^2 - 0.09^2\right)$

Solve to give

$a = 0.116 \text{ m}$

c To find the time from $O \to \dfrac{a}{2}$

$x = a\sin\omega t$

$\dfrac{a}{2} = a\sin 4.08t$

$\dfrac{1}{2} = \sin 4.08t$

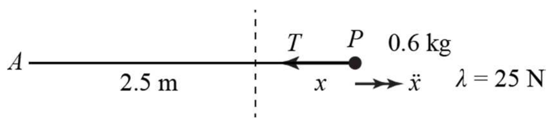
$4.08t = \dfrac{\pi}{6}$

$t = 0.128 \text{ s}$

Time for one oscillation is 1.540 s

Therefore the time required is $1.540 - 4 \times 0.1283 = 1.03 \text{ s}$

16



a $F = ma$

$-T = 0.6\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{25}{2.5}x = 10x$

$\therefore \quad 0.6\ddot{x} = -10x$

$\ddot{x} = -\dfrac{10}{0.6}x$

$\therefore$ S.H.M.

16 b $\omega^2 = \frac{10}{0.6}$

$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.6}{10}} = 1.539\ldots$

$\text{period} = 1.54\,\text{s}\,(3\,\text{s.f.})$

$\text{amplitude} = (4 - 2.5)\,\text{m} = 1.5\,\text{m}$

c $x = a\cos\omega t$

$x = 1.5\cos\left(\sqrt{\frac{10}{0.6}}t\right)$

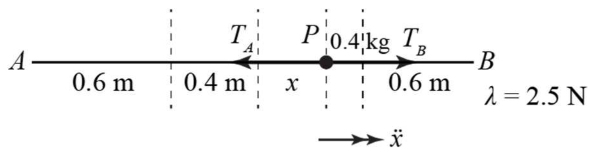
$x = -0.5\text{m} \quad -0.5 = 1.5\cos\left(\sqrt{\frac{10}{0.6}}t\right)$

$t = \sqrt{\frac{0.6}{10}}\cos^{-1}\left(-\frac{0.5}{1.5}\right) = 0.4680\ldots$

P takes 0.468s to move 2 m from B (3 s.f.).

B is an end-point.

17



a $F = ma$

$T_B - T_A = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T_A = \frac{2.5(0.4 + x)}{0.6}$

$T_B = \frac{2.5(0.4 - x)}{0.6}$

$\therefore \frac{2.5(0.4 - x)}{0.6} - \frac{2.5(0.4 + x)}{0.6} = 0.4\ddot{x}$

$-2 \times \frac{2.5x}{0.6} = 0.4\ddot{x}$

$\ddot{x} = -\frac{2 \times 2.5}{0.6 \times 0.4}x$

$\therefore$ S.H.M.

Consider P to be attached to two strings, each of natural length 0.6 m and modulus 2.5 N.

b $\omega^2 = \frac{2 \times 2.5}{0.6 \times 0.4} = \frac{5}{0.24}$

$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.24}{5}} = 1.376\ldots$

The period is 1.38s (3 s.f.)

17 c $x = a\cos\omega t$

At D, $x = 1 - 0.85 = 0.15$

$$0.15 = 0.3\cos\left(\sqrt{\frac{5}{0.24}}t\right)$$

$$\sqrt{\frac{5}{0.24}}t = \cos^{-1} 0.5$$

$$t = \sqrt{\frac{0.24}{5}}\cos^{-1} 0.5$$

$$t = 0.2294\ldots$$

P takes 0.229s (3 s.f.) to reach D.

For time from B (an end-point)

D and C are on the same side of the centre, so x is positive.

18 a

Hooke's law to AP:

$$T_1 = \frac{\lambda_1 x_1}{l_1} = \frac{20x_1}{5} = 4x_1$$

Hooke's law to BP:

$$T_2 = \frac{\lambda_2 x_2}{l_2} = \frac{18x_2}{3} = 6x_2$$

Resolving horizontally:

$T_1 = T_2$

So $4x_1 = 6x_2$

$$x_1 = \frac{3x_2}{2}$$

Length $AB = 12$

$5 + 3 + x_1 + x_2 = 12$

$x_1 + x_2 = 4$

$$\frac{3x_2}{2} + x_2 = 4$$

$$\frac{5x_2}{2} = 4$$

So $x_2 = \frac{8}{5}$ and $x_1 = \frac{12}{5}$

Hence the AP spring extension is 2.4 m and the PB spring extension is 1.6 m.

18 b Consider P displaced a distance x to the right of its equilibrium position.
Equation of motion of P:

$$T_2 - T_1 = m\ddot{x}$$
$$\frac{18(1.6-x)}{3} - \frac{20(2.4+x)}{5} = 0.4\ddot{x}$$
$$9.6 - 6x - 9.6 - 4x = 0.4\ddot{x}$$
$$-10x = 0.4\ddot{x}$$
$$\ddot{x} = -25x$$

Hence P oscillates with SHM.

c $\ddot{x} = -25x$

$$\omega^2 = 25 \Rightarrow \omega = 5$$

$$x = a\sin\omega t$$
$$x = a\sin 5t$$

Time from $x = 0$ to $x = 0.4$ is given by

$$0.4 = a\sin 5t$$
$$\frac{0.4}{a} = \sin 5t$$
$$t = \frac{1}{5}\arcsin\left(\frac{0.4}{a}\right)$$

So P stays within 0.4 m of the equilibrium position for $\frac{4}{5}\arcsin\left(\frac{0.4}{a}\right)$ seconds.

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$$

So $\frac{T}{3} = \frac{2\pi}{15}$

$$\therefore \frac{4}{5}\arcsin\left(\frac{0.4}{a}\right) = \frac{2\pi}{15}$$
$$\arcsin\left(\frac{0.4}{a}\right) = \frac{10\pi}{60} = \frac{\pi}{6}$$
$$\frac{0.4}{a} = 0.5$$
$$a = \frac{4}{5} = 0.8$$

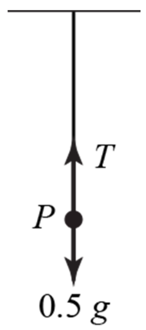
Initial speed of P

$= v_{max} = \omega a$ I

$= 5 \times \frac{4}{5}$

$= 4$ m s^{-1}.

19 a



In equilibrium:

$R(\uparrow)\, T = 0.5g$

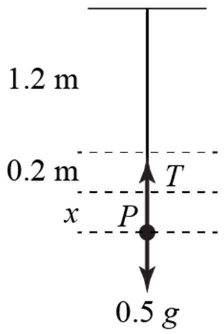
Hooke's Law : $T = \dfrac{\lambda x}{l}$

$0.5g = \dfrac{\lambda \times 0.2}{1.2}$

$\lambda = 0.5 \times \dfrac{1.2}{0.2}$

$\therefore \lambda = 3g \quad (\text{or } 29.4)$

b



For oscillations:

$F = ma$

$0.5g - T = 0.5\ddot{x}$

Hooke's Law : $T = \dfrac{\lambda x}{l}$

$T = \dfrac{3g(0.2 + x)}{1.2}$

$\therefore 0.5g - \dfrac{3g(0.2+x)}{1.2} = 0.5\ddot{x}$

$\ddot{x} = -\dfrac{3g}{0.5 \times 1.2}x = -5gx$

$\therefore$ S.H.M.

Of form $\ddot{x} = \omega^2 x$.

c $\omega^2 = 5g$

$\text{period} = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{5g}} = 0.8975...$

The period is 0.898s (3 s.f.).

From $\ddot{x} = -5gx$.

d String becomes slack when $x = -0.2$ m.

amplitude $= 0.35$ m

$v^2 = \omega^2(a^2 - x^2)$

$v^2 = 5g(0.35^2 - 0.2^2)$

$v = 2.010...$

The speed is 2.01 m s^{-1} (3 s.f.).

Use the exact value for ω^2.

e $v^2 = u^2 + 2as$

$0 = 2.010^2 - 2 \times 9.8s$

$s = \dfrac{2.010^2}{2 \times 9.8} = 0.2061...$

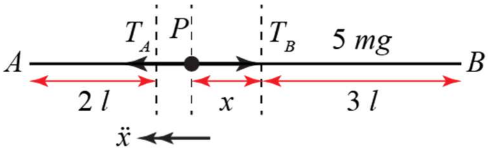
Distance above $O = 0.2 + 0.2061...$

$= 0.406\text{ m } (3\text{ s.f.})$

Once the string is slack the particle moves freely under gravity.

The particle is 0.2 m above O when the string becomes slack.

20



$\lambda = 5\,mg$

a Hooke's Law : $T = \frac{\lambda x}{l}$

$$T_A = \frac{5mg(l-x)}{2l}$$

$$T_B = \frac{5mg(l+x)}{2l}$$

Consider the particle to be attached to two strings, *AP* and *PB*, both with natural length $2l$ and modulus $5\,mg$.

$$F = ma$$

$$T_A - T_B = m\ddot{x}$$

$$\frac{5mg(l-x)}{2l} - \frac{5mg(l+x)}{2l} = m\ddot{x}$$

$$-\frac{5mgx}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{5gx}{l}$$

$\therefore$ S.H.M.

b $\omega^2 = \frac{5g}{l}$ period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{5g}}$

The period is $2\pi\sqrt{\left(\frac{l}{5g}\right)}$

c amplitude $= \frac{3l}{4}$

Find the amplitude from the given information.

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

$$v_{\max} = \omega a = \sqrt{\frac{5g}{l}} \times \frac{3l}{4} = \frac{3}{4}\sqrt{5gl}$$

Maximum speed when $x = 0$.

The maximum speed is $\frac{3}{4}\sqrt{5gl}$.

Challenge

a The equation of motion for mass m is given by

$$ma = -\frac{mMg}{(R+x)^2}$$

$$a = -\frac{Mg}{(R+x)^2}$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{Mg}{(R+x)^2}$$

The mass reaches its maximum height H when $v = 0$.
Separating the variables and integrating:

$$\int_u^0 v\mathrm{d}v = -\int_0^H \frac{MG}{(R+x)^2}\mathrm{d}x$$

$$\left[\frac{v^2}{2}\right]_u^0 = \left[\frac{MG}{(R+x)}\right]_0^H$$

$$-\frac{u^2}{2} = \frac{MG}{R+H} - \frac{MG}{R}$$

$$-\frac{u^2}{2} = MG\left(\frac{1}{R+H} - \frac{1}{R}\right)$$

$$-\frac{u^2}{2} = MG\left(\frac{R-(R+H)}{R(R+H)}\right)$$

$$-\frac{u^2}{2} = -MG\left(\frac{H}{R(R+H)}\right)$$

$$\frac{u^2}{2} = MG\left(\frac{H}{R(R+H)}\right)$$

$$2MGH = u^2\left(R^2 + RH\right)$$

$$2MGH - RHu^2 = u^2R^2$$

$$H\left(2MG - Ru^2\right) = u^2R^2$$

$$H = \frac{u^2R^2}{2MG - Ru^2}$$

$$H = \frac{Ru^2}{\frac{2MG}{R} - u^2}$$

b As $u^2 \to \frac{2MG}{R}$, $H \to \infty$

The escape velocity is therefore given by

$$u = \sqrt{\frac{2MG}{R}}$$

$$u = \sqrt{\frac{2\times 5.98\times 10^{24}\times 6.7\times 10^{-11}}{6.4\times 10^6}}$$

$u = 1.12\times 10^4$ m s^{-1}.

