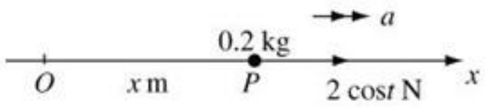
Dynamics 5A

1



a $F = ma$

$2\cos t = 0.2a$

$0.2\frac{dv}{dt} = 2\cos t$ ← Force is a function of time so use $a = \frac{dv}{dt}$.

$v = \frac{2}{0.2}\int \cos t\, dt$ ← Integrate to obtain an expression for v.

$v = 10\sin t + c$ ← Don't forget the constant.

$t = 0 \quad v = 0$

$0 = 0 + c \therefore c = 0$

$v = 10\sin t$

$t = 2 \quad v = 10\sin 2 = 9.092...$

When $t = 2$ the speed of P is 9.09 m s^{-1} (3 s.f.)

b $t = 3 \quad v = 10\sin 3 = 1.411...$

When $t = 3$ the speed of P is 1.41 m s^{-1} (3 s.f.)

c $v = 0 \quad 0 = 10\sin t$ ← P comes to rest when $v = 0$.

$\sin t = 0$

$t = 0, \pi, ...$

P first comes to rest when $t = \pi$. ← Exact answers are best.

d $v = 10\sin t$

$\frac{dx}{dt} = 10\sin t$

$x = 10\int \sin t\, dt$ ← Integrate to obtain an expression for x.

$x = -10\cos t + K$

$t = 0, x = 0 \quad 0 = -10 + K \therefore K = 10$

$x = -10\cos t + 10$

$t = 2 \quad x = -10\cos t + 10 = 14.16...$

When $t = 2$ $OP = 14.2$ m (3s.f.)

e $t = \pi \quad x = -10\cos\pi + 10$

$= 10 + 10 = 20$

When P comes to rest $OP = 20$ m.

2 a $F = ma$

$$\frac{60\,000}{(t+5)^2} = 1200a$$

$$a = \frac{50}{(t+5)^2}$$

$$\frac{dv}{dt} = \frac{50}{(t+5)^2}$$

Force is a function of time so use $a = \frac{dv}{dt}$.

$$v = \int \frac{50}{(t+5)^2}\,dt$$

Integrate to obtain an expression for v.

$$v = -\frac{50}{(t+5)} + c$$

$$t = 0,\ v = 0 \quad \therefore 0 = -\frac{50}{5} + c$$

$$c = 10$$

$$v = -\frac{50}{t+5} + 10$$

As $t \to \infty$ $-\frac{50}{t+5} + 0$

$\therefore V = 10$

b $v = -\frac{50}{t+5} + 10$

$$\frac{dx}{dt} = -\frac{50}{t+5} + 10$$

$$x = -50\ln(t+5) + 10t + K$$

$$t = 0, x = 0 \qquad 0 = -50\ln 5 + K$$

$$K = 50\ln 5$$

$$\therefore x = -50\ln(t+5) + 10t + 50\ln 5$$

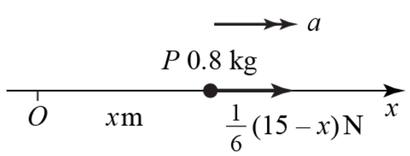
$$t = 4 \quad x = -50\ln 9 + 40 + 50\ln 5$$

$$x = 40 + 50\ln\frac{5}{9}$$

$$x = 10.61...$$

The van moves 10.6 m in the first 4 seconds (3 s.f.)

3



a Maximum speed $\Rightarrow$ acceleration zero $\Rightarrow$ force is zero

$$\therefore \frac{1}{6}(15 - x) = 0 \quad \therefore x = 15$$

3 b $F = ma$

$\frac{1}{6}(15-x) = 0.8a$

$a = \frac{1}{4.8}(15-x)$

$v\frac{dv}{dx} = \frac{1}{4.8}(15-x)$ ← Force is a function of x so use $a = v\frac{dv}{dx}$.

$\int v\,dv = \frac{1}{4.8}\int(15-x)\,dx$ ← Separate the variables.

$\frac{1}{2}v^2 = \frac{1}{4.8}\left(15x - \frac{1}{2}x^2\right) + c$

$x = 15, v = 12$ ← a tells you the initial conditions.

$\frac{1}{2}\times 12^2 = \frac{1}{4.8}\left(15\times 15 - \frac{1}{2}\times 15^2\right) + c$

$c = \frac{1}{2}\times 12^2 - \frac{1}{4.8}\times\frac{1}{2}\times 15^2$

$c = 48.5625$

$\frac{1}{2}v^2 = \frac{1}{4.8}\left(15x - \frac{1}{2}x^2\right) + 48.5625$

$t = 0, x = 0 \quad v^2 = 2\times 48.5625$ ← P is at 0 when $t = 0$.

$v = 9.855$

When $t = 0$ P's speed is 9.86 m s^{-1} (3 s.f.)

4 a $0.75v\frac{dv}{dx} = 2e^{-x} + 2$

$v\frac{dv}{dx} = \frac{8}{3}e^{-x} + \frac{8}{3}$

Separating the variables and integrating:

$\int v\,dv = \int\left(\frac{8}{3}e^{-x} + \frac{8}{3}\right)dx + c$

$\frac{v^2}{2} = -\frac{8}{3}e^{-x} + \frac{8x}{3} + c$

At $x = 0$, $v = 5$:

$\frac{5^2}{2} = -\frac{8}{3}e^{-0} + \frac{8\times 0}{3} + c$

$c = \frac{91}{6}$

$\frac{v^2}{2} = -\frac{8}{3}e^{-x} + \frac{8x}{3} + \frac{91}{6}$

At $x = 3$

$\frac{v^2}{2} = -\frac{8}{3}e^{-3} + 8 + \frac{91}{6}$

$\Rightarrow v = 6.79\text{ m s}^{-1}$

4 b At $x = 7$

$$\frac{v^2}{2} = -\frac{8}{3}e^{-7} + \frac{56}{3} + \frac{91}{6}$$

$$\Rightarrow v = 8.23\,\text{m s}^{-1}$$

c Work done $= \int_3^7 F\,\mathrm{d}x$

$$= \int_3^7 \left(2e^{-x} + 2\right)\,\mathrm{d}x$$

$$= \left[-2e^{-x} + 2x\right]_3^7$$

$$= -2e^{-7} + 14 - \left(-2e^{-3} + 6\right)$$

$$= 8.10\text{ J}$$

5 $F = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$

$$\frac{3}{x+2} = \frac{1}{2}v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables and integrating:

$$\int_0^X \frac{3}{x+2}\,\mathrm{d}x = \int_{1.5}^2 \frac{v}{2}\,\mathrm{d}v$$

$$3\ln(x+2) = \frac{v^2}{4} + c$$

$$t = 0,\ v = 1.5 \quad \therefore 3\ln(2) = \frac{1.5^2}{4} + c$$

$$c = 1.517...$$

When $v = 2$

$$\ln(x+2) = \frac{2^2}{12} + \frac{1.517}{3} = 0.83898...$$

$$x = e^{0.83898} - 2$$

$$x = 0.314$$

6 a $m\frac{\mathrm{d}v}{\mathrm{d}t}=F$

$\frac{1}{4}\frac{\mathrm{d}v}{\mathrm{d}t}=-\frac{8}{(t+1)^2}$

$\frac{\mathrm{d}v}{\mathrm{d}t}=-\frac{32}{(t+1)^2}$

Integrating gives:

$v=\frac{32}{t+1}+c$

At $t=0,\ v=10$

$10=32+c$

$c=-22$

$v=\frac{32}{t+1}-22$

$v=2\left(\frac{16}{(t+1)}-11\right)$

b $x=\int v\ \mathrm{d}t=\int\left(\frac{32}{t+1}-22\right)\mathrm{d}t$

$x=32\ln(t+1)-22t+c$

At $t=0,\ x=0$, so $c=0$

$x=32\ln(t+1)-22t$

When $t=5$:

$x=32\ln 6-22\times 5$

$x=32\ln 6-110$

7 F is directed towards O, so

$$F = -\frac{k}{(x+2)^2}$$

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{k}{(x+2)^2}$$

$$\frac{3}{5}v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{k}{(x+2)^2}$$

Separating the variables and integrating:

$$\int \frac{3v}{5}\,\mathrm{d}v = -\int \frac{k}{(x+2)^2}$$

$$\frac{3v^2}{10} = \frac{k}{(x+2)} + c$$

At $x = 3,\ v^2 = 25$:

$$7.5 = \frac{k}{5} + c$$

$$37.5 = k + 5c \qquad \textbf{(1)}$$

At $x = 8,\ v^2 = 3$:

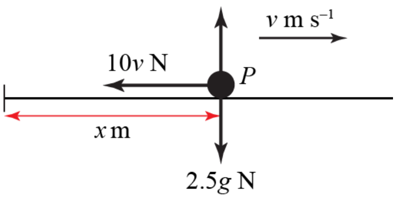
$$0.9 = \frac{k}{10} + c$$

$$9 = k + 10c \qquad \textbf{(2)}$$

Solving equations **(1)** and **(2)** simultaneously gives:
$5c = -28.5$
So $c = -5.7$

Therefore $k = 66$

8



$R(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$

$$-10v = 2.5\frac{dv}{dt}$$

Separating the variables

$$\int 4\ dt = -\int \frac{1}{v}dv$$

$$4t = A - \ln v$$

When $t = 0, v = 24$

$0 = A - \ln 24 \Rightarrow A = \ln 24$

Hence

$4t = \ln 24 - \ln v$

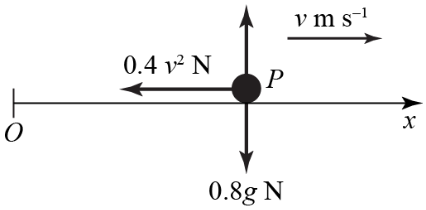
$$t = \frac{1}{4}\ln\left(\frac{24}{v}\right)$$

When $v = 6$

$$t = \frac{1}{4}\ln 4 (\approx 0.347)$$

P takes $\frac{1}{4}\ln 4s (= 0.347s, 3\text{d.p.})$ to slow from 24 m s^{-1} to 6 m s^{-1}.

9



$R(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$-0.4v^2 = 0.8v\frac{dv}{dx}$$

Separating the variables

$$\int 1 \, dx = -2\int \frac{1}{v} dv$$

$$x = A - 2\ln v$$

At $x = 0, v = 12$

$0 = A - 2\ln 12 \Rightarrow A = 2\ln 12$

Hence

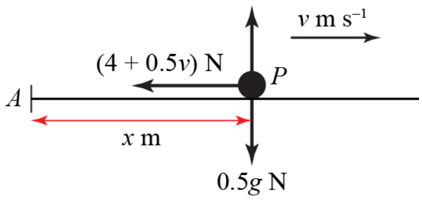
$$x = 2\ln 12 - 2\ln v = 2\ln\left(\frac{12}{v}\right)$$

When $v = 6$

$x = 2\ln 2$

The distance P moves before its speed is halved is $2\ln 2\,\text{m} = 1.39\,\text{m}$ (3 s.f.)

10 a



$\mathrm{R}(\rightarrow) \qquad \mathbf{F} = m\mathbf{a}$

$$-(4+0.5v) = 0.5\frac{dv}{dt}$$

Separating the variables

$$\int 1\ dt = -\int \frac{1}{8+v}\,dv$$

$$t = A - \ln(8+v)$$

When $t = 0, v = 12$

$0 = A - \ln 20 \Rightarrow A = \ln 20$

Hence

$$t = \ln 20 - \ln(8+v) = \ln\left(\frac{20}{8+v}\right)$$

When $v = 0$

$$t = \ln\left(\frac{20}{8}\right) = \ln 2.5$$

The time taken for P to move from A to B is in 2.5 s = 0.916 s (3 d.p.).

b $\mathrm{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$

$$-(4+0.5v) = 0.5v\frac{dv}{dx}$$

Separating the variables

$$\int 1\ dx = -\int \frac{v}{8+v}\,dv$$

$$\frac{v}{8+v} = \frac{8+v-8}{8+v} = 1 - \frac{8}{8+v}$$

Hence

$$\int 1\ dx = -\int\left(1 - \frac{8}{8+v}\right)dv$$

$$x = A - v + 8\ln(8+v)$$

At $x = 0, v = 12$

$0 = A - 12 + 8\ln 20 \Rightarrow A = 12 - 8\ln 20$

Hence $\quad x = 12 - v - (8\ln 20 - 8\ln(8+v))$

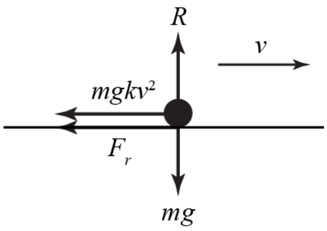
$$= 12 - v - 8\ln\left(\frac{20}{8+v}\right)$$

When $v = 0$

$x = 12 - 8\ln 2.5$

$AB = (12 - 8\ln 2.5)\,\mathrm{m} = 4.67\,\mathrm{m} \quad$ (3s.f.)

11



$R(\uparrow)\quad R = mg$

As friction is limiting

$$F_r = \mu R = \mu mg$$

$R(\rightarrow)\qquad \mathbf{F} = m\mathbf{a}$

$$-F_r - kmgv^2 = ma$$

$$-\mu \not{m} g - k \not{m} g v^2 = \not{m} v \frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int g\,\mathrm{d}x = -\int \frac{v}{\mu + kv^2}\,\mathrm{d}v$$

$$gx = A - \frac{1}{2k}\ln\left(\mu + kv^2\right)$$

At $x = 0, v = u$

$$0 = A - \frac{1}{2k}\ln\left(\mu + ku^2\right) \Rightarrow A = \frac{1}{2k}\ln\left(\mu + ku^2\right)$$

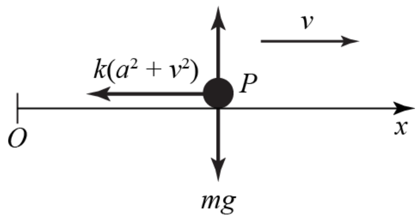
Hence

$$x = \frac{1}{2kg}\left(\ln\left(\mu + ku^2\right) - \ln\left(\mu + kv^2\right)\right) = \frac{1}{2kg}\ln\left(\frac{\mu + ku^2}{\mu + kv^2}\right)$$

When $v = 0$

$$x = \frac{1}{2kg}\ln\left(\frac{\mu + ku^2}{\mu}\right)$$

12 a



$R(\rightarrow)$ $\qquad$ $\mathbf{F} = m\mathbf{a}$

$$-k\left(a^2+v^2\right)=m\frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\int 1\,\mathrm{d}t=-\frac{m}{k}\int\frac{1}{a^2+v^2}\,\mathrm{d}v$$

$$t=A-\frac{m}{ak}\arctan\left(\frac{v}{a}\right)$$

When $t=0, v=U$

$$0=A-\frac{m}{ak}\arctan\left(\frac{U}{a}\right)\Rightarrow A=\frac{m}{ak}\arctan\left(\frac{U}{a}\right)$$

Hence

$$t=\frac{m}{ak}\arctan\left(\frac{U}{a}\right)-\frac{m}{ak}\arctan\left(\frac{v}{a}\right)$$

When $t=T, v=\frac{1}{2}U$

$$T=\frac{m}{ak}\arctan\left(\frac{U}{a}\right)-\frac{m}{ak}\arctan\left(\frac{\frac{1}{2}U}{a}\right)$$

$$T=\frac{m}{ak}\left[\arctan\left(\frac{U}{a}\right)-\arctan\left(\frac{U}{2a}\right)\right],\quad \text{as required}$$

b $R(\rightarrow)$ $\qquad$ $\mathbf{F} = m\mathbf{a}$

$$-k\left(a^2+v^2\right)=mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x=-\frac{m}{k}\int\frac{v}{a^2+v^2}\,\mathrm{d}v$$

$$x=A-\frac{m}{2k}\ln\left(a^2+v^2\right)$$

When $x=0, v=U$

$$0=A-\frac{m}{2k}\ln\left(a^2+U^2\right)\Rightarrow A=\frac{m}{2k}\ln\left(a^2+U^2\right)$$

12 b continued

So $x = \frac{m}{2k}\ln\left(a^2 + U^2\right) - \frac{m}{2k}\ln\left(a^2 + v^2\right)$

$$= \frac{m}{2k}\ln\left(\frac{a^2 + U^2}{a^2 + v^2}\right)$$

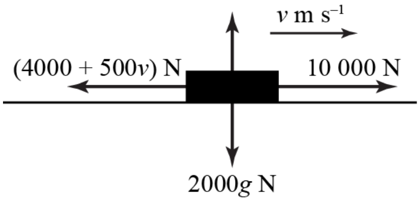
When $v = \frac{U}{2}$:

$$x = \frac{m}{2k}\ln\left(\frac{a^2 + U^2}{a^2 + \left(\frac{U}{2}\right)^2}\right)$$

$$= \frac{m}{2k}\ln\left(\frac{a^2 + U^2}{a^2 + \frac{U^2}{4}}\right)$$

$$= \frac{m}{2k}\ln\left(\frac{4a^2 + 4U^2}{4a^2 + U^2}\right)$$

13 a



$R(\rightarrow) \qquad \mathbf{F} = m\mathbf{a}$

$$10\,000 - (4000 + 500v) = 2000a$$

$$6000 - 500v = 2000\frac{dv}{dt}$$

Dividing throughout by 500

$$12 - v = 4\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = 4\int \frac{1}{12 - v}\,dv$$

$$t = A - 4\ln(12 - v)$$

$$\ln(12 - v) = B - \frac{t}{4}, \text{ where } B = \frac{1}{4}A$$

$$12 - v = e^{B - \frac{t}{4}} = e^B e^{\frac{t}{4}} = Ce^{-\frac{t}{4}}, \text{ where } C = e^B$$

Hence

$v = 12 - Ce^{-\frac{t}{4}}$

When $t = 0, v = 0$

$0 = 12 - C \Rightarrow C = 12$

Hence

$v = 12\left(1 - e^{-\frac{t}{4}}\right)$

b As $t \to \infty, e^{-\frac{t}{4}} \to 0$ and $v \to 12$

The terminal speed of the lorry is 12 m s^{-1}.

14 a $F = ma$

$$-\frac{mgv}{k} - mg = m\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\frac{gv}{k} + g = -\frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables and integrating:

$$-\int_U^0 \frac{k}{g(v+k)} = \int_0^T \mathrm{d}t$$

$$-\left[\frac{k}{g}\ln|v+k|\right]_U^0 = [t]_0^T$$

$$-\frac{k}{g}\ln|k| - \left(-\frac{k}{g}\ln|k+U|\right) = T$$

$$T = \frac{k}{g}\ln\left(\frac{k+U}{k}\right)$$

b $$-\frac{mgv}{k} - mg = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{gv}{k} + g = -v\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{gv+gk}{k} = -v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables before integrating:

$$-\int_U^0 \frac{kv\ \mathrm{d}v}{gv+gk} = \int_0^H \mathrm{d}x$$

$$-\frac{k}{g}\int_U^0 \frac{v\ \mathrm{d}v}{v+k} = \int_0^H \mathrm{d}x$$

$$-\frac{k}{g}\int_U^0 \left(1 - \frac{k}{v+k}\right)\mathrm{d}v = \int_0^H \mathrm{d}x$$

$$-\frac{k}{g}\left[v - k\ln|v+k|\right]_U^0 = [x]_0^H$$

$$H = \frac{k}{g}\left[v - k\ln|v+k|\right]_0^U$$

$$H = \frac{k}{g}\left[\left(U - k\ln(U+k)\right) - \left(0 - k\ln k\right)\right]$$

$$H = \frac{k}{g}\left[U - k\ln(U+k) + k\ln k\right]$$

$$H = \frac{k}{g}\left[U - k\ln\left(\frac{k+U}{k}\right)\right]$$

15 a $F = ma$

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = mg - mgkv^2$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = g\left(1 - kv^2\right)$$

Separating the variables and integrating:

$$\int \frac{v\ dv}{g\left(1-kv^2\right)} = \int \mathrm{d}x + c$$

$$-\frac{1}{2kg}\ln\left|1-kv^2\right| = x + c$$

At $x = 0,\ v = 0$. So $c = 0$

$$-\frac{1}{2kg}\ln\left|1-kv^2\right| = x$$

$$\ln\left|1-kv^2\right| = -2kgx$$

$$1 - kv^2 = e^{-2kgx}$$

$$kv^2 = 1 - e^{-2kgx}$$

$$v^2 = \frac{1-e^{-2kgx}}{k}$$

$$v = \sqrt{\frac{1-e^{-2kgx}}{k}}$$

b As $x \to \infty$, $v \to \sqrt{\frac{1}{k}}$,

so the terminal velocity

is $\sqrt{\frac{1}{k}}$ m s^{-1}.

c This model has the particle rapidly approaching terminal velocity; within two metres of release the exponential term is of the order 10^{-19}.

Challenge

a Work done $= \int_a^b \left(3x^2 - x^{\frac{1}{3}}\right) \mathrm{d}x$

$$= \left[x^3 - \frac{3}{4} x^{\frac{4}{3}} \right]_a^b$$

$$= b^3 - \frac{3}{4} b^{\frac{4}{3}} - a^3 + \frac{3}{4} a^{\frac{4}{3}}$$

Hence the work done is independent of the initial velocity.

b Work done $= \int_0^6 \left(3x^2 - x^{\frac{1}{3}}\right) \mathrm{d}x$

$$= \left[x^3 - \frac{3}{4} x^{\frac{4}{3}} \right]_0^6$$

$$= 6^3 - \frac{3}{4} \times 6^{\frac{4}{3}}$$

$$= 208 \text{ J}$$