Kinematics 4C

1 a $a = \frac{dv}{dt} = e^{-v}$

Separating the variables and integrating

$\int e^{v} dv = \int 1 dt$

$e^{v} = t + C$

When $t = 0, \ v = 0$

$1 = 0 + C \Rightarrow C = 1$

So $e^{v} = t + 1 \Rightarrow v = \ln(t+1)$

b When $t = 10, \ v = \ln(10+1) = \ln 11$

2 $a = \frac{dv}{dt} = -8v$

Separating the variables and integrating

$\int \frac{1}{v} dv = -\int 8 dt$

$\ln v = -8t + C$

When $v = 18, t_1 = \frac{1}{8}(C - \ln 18)$

When $v = 6, t_2 = \frac{1}{8}(C - \ln 6)$

Required time is $t_2 - t_1$

$t_2 - t_1 = \frac{1}{8}(C - \ln 6) - \frac{1}{8}(C - \ln 18) = \frac{1}{8}(\ln 18 - \ln 6) = \frac{1}{8}\ln\left(\frac{18}{6}\right) = \frac{1}{8}\ln 3 = 0.137 \text{ s (3 s.f.)}$

3 a $a = \frac{dv}{dt} = -(3 + 0.6v)$

Separating the variables and integrating

$\int \frac{1}{3+0.6v} dv = -\int 1 dt$

$\frac{1}{0.6}\ln(3+0.6v) = -t + C$

When $t = 0, \ v = 12$

$\frac{1}{0.6}\ln 10.2 = 0 + C \Rightarrow C = \frac{1}{0.6}\ln 10.2$

So $\frac{1}{0.6}\ln(3+0.6v) = -t + \frac{1}{0.6}\ln 10.2$

When $v = 0$

$\frac{1}{0.6}\ln 3 = -t + \frac{1}{0.6}\ln 10.2$

$\Rightarrow t = \frac{1}{0.6}(\ln 10.2 - \ln 3) = \frac{1}{0.6}\ln\left(\frac{10.2}{3}\right) = \frac{1}{0.6}\ln 3.4 = 2.04 \text{ s (3 s.f.)}$

3 b $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x} = -(3+0.6v)$

Separating the variables

$$\int \frac{v}{3+0.6v}\mathrm{d}v = -\int 1\mathrm{d}x$$

To integrate, rearrange the expression in v

$$\frac{v}{3+0.6v} = \frac{5v}{15+3v} = \frac{5}{3} - \frac{25}{15+3v}$$

So integrating gives

$$\frac{5v}{3} - \frac{25}{3}\ln(15+3v) = -x + D$$

At $x = 0$ (point A), $v = 12$ so

$$D = 20 - \frac{25}{3}\ln 51$$

When $v = 0$ (point B)

$$-\frac{25}{3}\ln 15 = -x + 20 - \frac{25}{3}\ln 51$$

$$x = 20 - \frac{25}{3}\ln\left(\frac{51}{15}\right) = 20 - \frac{25}{3}\ln 3.4 = 9.80\,\text{m (3 s.f.)}$$

4 a $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = g - 2v$

Separating the variables and integrating

$$\int \frac{1}{g-2v}\mathrm{d}v = \int 1\mathrm{d}t$$

$$-\frac{1}{2}\ln(g-2v) = t + C$$

When $t = 0,\ v = 0$

$$-\frac{1}{2}\ln g = 0 + C \Rightarrow C = -\frac{1}{2}\ln g$$

So $-\dfrac{1}{2}\ln(g-2v) = t - \dfrac{1}{2}\ln g$

$$\Rightarrow \ln(g-2v) - \ln g = -2t$$

$$\Rightarrow \frac{g-2v}{g} = \mathrm{e}^{-2t}$$

$$\Rightarrow 2v = g - g\mathrm{e}^{-2t} = g(1-\mathrm{e}^{-2t})$$

4 b $v = \frac{dx}{dt}$, so from part **a** $2\frac{dx}{dt} = g(1 - e^{-2t})$

Separating the variables and integrating

$$\int 2dx = \int g(1 - e^{-2t})dt$$

$$2x = gt + \frac{g}{2}e^{-2t} + D$$

When $t = 0$, $x = 0$

$$0 = 0 + \frac{g}{2} + D \Rightarrow D = -\frac{g}{2}$$

So $x = \frac{g}{2}t + \frac{g}{4}e^{-2t} - \frac{g}{4}$

When $t = 2$

$$x = g + \frac{g}{4}e^{-4} - \frac{g}{4} = \frac{g}{4}(3 + e^{-4})$$

5 a $a = \frac{dv}{dt} = 3 - 0.25v$

Separating the variables and integrating

$$\int \frac{1}{3 - 0.25v}dv = \int 1dt$$

$$-4\ln(3 - 0.25v) = t + C$$

When $t = 0$, $v = 0$

$$-4\ln 3 = 0 + C \Rightarrow C = -4\ln 3$$

So $-4\ln(3 - 0.25v) = t - 4\ln 3$

$$\Rightarrow \ln\left(\frac{3 - 0.25v}{3}\right) = -0.25t$$

$$\Rightarrow \frac{3 - 0.25v}{3} = e^{-0.25t}$$

$$\Rightarrow v = 12 - 12e^{-0.25t} = 12(1 - e^{-0.25t})$$

This is equivalent to the textbook solution $v = 12 - \frac{12}{e^{\frac{t}{4}}}$

b As $t \to \infty$, $e^{-0.25t} \to 0$, so $v = 12(1 - e^{-0.25t}) \to 12$, therefore $v_{max} = 12\,m\,s^{-1}$

6 $a = v\dfrac{dv}{dx} = 0.6v^2$

Separating the variables and integrating

$$\int \frac{1}{0.6v}\,dv = \int 1\,dx$$

$$\frac{1}{0.6}\ln v = x + C$$

At $x = 0,\ v = 14$

$$\frac{1}{0.6}\ln 14 = 0 + C \Rightarrow C = \frac{1}{0.6}\ln 14$$

So when $v = 28$

$$\frac{1}{0.6}\ln 28 = x + \frac{1}{0.6}\ln 14$$

$$\Rightarrow x = \frac{1}{0.6}(\ln 28 - \ln 14) = \frac{1}{0.6}\ln\left(\frac{28}{14}\right) = \frac{1}{0.6}\ln 2 = 1.16\text{ m (3 s.f.)}$$

7 $a = v\dfrac{dv}{dx} = -(k + v^2)$

Separating the variables and integrating

$$\int \frac{v}{(k+v^2)}\,dv = -\int 1\,dx$$

$$\frac{1}{2}\ln(k+v^2) = -x + C$$

At $x = 0,\ v = u$

$$\frac{1}{2}\ln(k+u^2) = 0 + C \Rightarrow C = \frac{1}{2}\ln(k+u^2)$$

So when $v = 0$

$$\frac{1}{2}\ln k = -x + \frac{1}{2}\ln(k+u^2)$$

$$\Rightarrow x = \frac{1}{2}\ln(k+u^2) - \frac{1}{2}\ln k = \frac{1}{2}\ln\left(\frac{k+u^2}{k}\right)\text{ m}$$

8 a $a = \frac{dv}{dt} = -(a^2 + v^2)$

Separating the variables and integrating

$\int \frac{1}{a^2 + v^2} dv = -\int 1 dt$

$\frac{1}{a}\arctan\frac{v}{a} = -t + C$

When $t = 0$, $v = U$

$C = \frac{1}{a}\arctan\frac{U}{a}$

When $t = T$, $v = \frac{1}{2}U$

$\frac{1}{a}\arctan\frac{U}{2a} = -T + \frac{1}{a}\arctan\frac{U}{a}$

$\Rightarrow T = \frac{1}{a}\left(\arctan\frac{U}{a} - \arctan\frac{U}{2a}\right)$ as required.

b $a = v\frac{dv}{dx} = -(a^2 + v^2)$

Separating the variables and integrating

$\int \frac{v}{(a^2 + v^2)} dv = -\int 1\, dx$

$\frac{1}{2}\ln(a^2 + v^2) = -x + D$

When $v = U$, $x = 0$

$\frac{1}{2}\ln(a^2 + U^2) = 0 + D \Rightarrow D = \frac{1}{2}\ln(a^2 + U^2)$

So when $v = \frac{1}{2}U$

$\frac{1}{2}\ln\left(a^2 + \frac{U^2}{4}\right) = -x + \frac{1}{2}\ln(a^2 + U^2)$

$\Rightarrow x = \frac{1}{2}\left(\ln(a^2 + U^2) - \ln\left(a^2 + \frac{U^2}{4}\right)\right) = \frac{1}{2}\ln\left(\frac{a^2 + U^2}{a^2 + \frac{U^2}{4}}\right) = \frac{1}{2}\ln\left(\frac{4(a^2 + U^2)}{4a^2 + U^2}\right)$ m

9 $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1600 - v^2}{64v}$

Separating the variables and integrating

$\int \frac{64v}{1600 - v^2}\mathrm{d}v = \int 1\,\mathrm{d}t$

$-32\ln(1600 - v^2) = t + C$

$t = C - 32\ln(1600 - v^2)$

So $T = C - 32\ln(1600 - 20^2) - (C - 32\ln(1600 - 10^2)) = 32(\ln 1500 - \ln 1200) = 32\ln\frac{1500}{1200} = 32\ln\frac{5}{4}$