Kinematics 4A

1 Let the velocity of P at time t be $v\,\text{m s}^{-1}$

$v = \int a\,\text{d}t = \int 3\text{e}^{-0.25t}\text{d}t = -12\text{e}^{-0.25t} + C$

When $t = 0$, $v = 4$

$4 = -12 + C \Rightarrow C = 16$

$v = 16 - 12\text{e}^{-0.25t}$

The velocity of the particle at time t seconds is $(16 - 12\text{e}^{-0.25t})\,\text{m s}^{-1}$

2 Let the displacement of P from O at time t be x m.

$x = \int v\,\text{d}t = \int t\sin t\,\text{d}t$

Using integration by parts

$x = -t\cos t + \int \cos t\,\text{d}t = -t\cos t + \sin t + C$

When $t = 0$, $x = 0$

$0 = 0 + 0 + C \Rightarrow C = 0$

$x = -t\cos t + \sin t$

When $t = \frac{\pi}{2}$

$x = -\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 1$

Hence P is 1 metre from O, as required.

3 Let the displacement of P from point A at time t be s m.

$s = \int v\,\text{d}t = \int \frac{4}{3+2t}\text{d}t = 2\ln(3+2t) + C$

When $t = 0$, $s = 0$

$0 = 2\ln 3 + C \Rightarrow C = -2\ln 3$

$s = 2\ln(3+2t) - 2\ln 3 = 2\ln\left(\frac{3+2t}{3}\right)$

When $t = 3$

$s = 2\ln\left(\frac{3+6}{3}\right) = 2\ln 3$

So $AB = 2\ln 3\,\text{m}$

4 Let the velocity of P at time t be $v\,\mathrm{m\,s^{-1}}$

$$v=\int a\,\mathrm{d}t=\int 4\mathrm{e}^{\frac{1}{2}t}\,\mathrm{d}t=8\mathrm{e}^{\frac{1}{2}t}+C$$

When $t=0$, $v=0$

$$0=8+C\Rightarrow C=-8$$

$$v=8\mathrm{e}^{\frac{1}{2}t}-8$$

The distance moved in the interval $0\leqslant t\leqslant 2$ is given by

$$s=\int_0^2 v\,\mathrm{d}t=\int_0^2 8\mathrm{e}^{\frac{1}{2}t}-8\,\mathrm{d}t$$

$$=\left[16\mathrm{e}^{\frac{1}{2}t}-8t\right]_0^2=(16\mathrm{e}^1-16)-16$$

$$=16\mathrm{e}-32=11.5\text{ (3 s.f.)}$$

The distance moved is 11.5 m (3 s.f.).

5 a Let the acceleration of P at time t be $a\,\mathrm{m\,s^{-2}}$

$$v=4\cos 3t$$

So $a=\dfrac{\mathrm{d}v}{\mathrm{d}t}=-12\sin 3t$

When $t=\dfrac{\pi}{12}$

$$a=-12\sin\frac{\pi}{4}=-12\times\frac{1}{\sqrt{2}}=-6\sqrt{2}$$

The magnitude of the acceleration when $t=\dfrac{\pi}{12}$ is $6\sqrt{2}\,\mathrm{m\,s^{-2}}$

b $x=\int v\,\mathrm{d}t=\int 4\cos 3t\,\mathrm{d}t=\dfrac{4}{3}\sin 3t+C$

When $t=0$, $x=0$

$$0=\frac{4}{3}\times 0+C\Rightarrow C=0$$

So $x=\dfrac{4}{3}\sin 3t$

c When P is at O, $x=0$

$$x=\frac{4}{3}\sin 3t=0\Rightarrow \sin 3t=0$$

The smallest positive value of t is given by $3t=\pi\Rightarrow t=\dfrac{\pi}{3}$

6 $v = \int a\,\mathrm{d}t = \int \frac{6t}{(2+t^2)^2}\,\mathrm{d}t$

Using integration by substitution, let $u = 2+t^2$, so $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$

$$v = \int \frac{6t}{(2+t^2)^2}\,\mathrm{d}t = \int \frac{3}{(2+t^2)^2} \times 2t\,\mathrm{d}t$$

$$= \int \frac{3}{u^2}\,\mathrm{d}u = \int 3u^{-2}\,\mathrm{d}u$$

$$= -3u^{-1} + C = C - \frac{3}{u}$$

$$= C - \frac{3}{2+t^2}$$

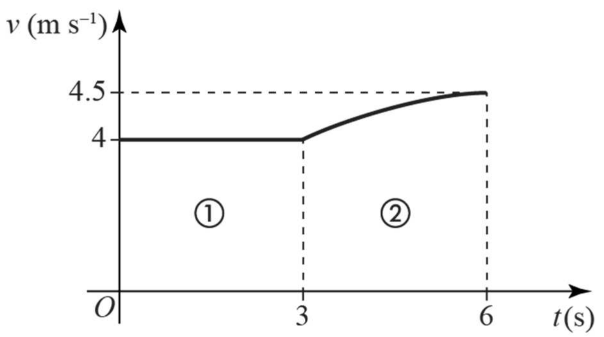
When $t = 0$, $v = 0$

$$0 = C - \frac{3}{2} \Rightarrow C = \frac{3}{2}$$

So $v = \frac{3}{2} - \frac{3}{2+t^2}$

7 a For $v = 4$, the graph is a straight line from (0, 4) to (3, 4).

For $v = 5 - \frac{3}{t}$, the graph is part of a reciprocal curve joining (3, 4) to (6, 0.5)



b The distance moved in the first three seconds is represented by the area labelled (1).
Let this area be A_1. Then $A_1 = 3 \times 4 = 12$

The distance travelled in the next three seconds is represented by the area labelled (2).
Let this area be A_2.

$$A_2 = \int_3^6 \left(5 - \frac{3}{t}\right)\mathrm{d}t = \left[5t - 3\ln t\right]_3^6 = (30 - 3\ln 6) - (15 - 3\ln 3)$$

$$= 15 - 3(\ln 6 - \ln 3) = 15 - 3\ln 2$$

So the displacement of P from O when $t = 6$ is $(12 + 15 - 3\ln 2)\,\text{m} = (27 - 3\ln 2)\,\text{m}$.

8 a $v = \int a\,\mathrm{d}t = \int \sin\frac{1}{2}t\,\mathrm{d}t = -2\cos\frac{1}{2}t + C$

When $t = 0$, $v = 0$

$0 = -2 + C \Rightarrow C = 2$

$v = 2 - 2\cos\frac{1}{2}t$

When $t = 2\pi$

$v = 2 - 2\cos\pi = 2 - (2\times -1) = 4$

The speed of P when $t = 2\pi$ is $4\,\mathrm{m\,s^{-1}}$

b $x = \int v\,\mathrm{d}t = \int\left(2 - 2\cos\frac{1}{2}t\right)\mathrm{d}t = 2t - 4\sin\frac{1}{2}t + B$

When $t = 0$, $x = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

$x = 2t - 4\sin\frac{1}{2}t$

When $t = \frac{\pi}{2}$

$x = 2\times\frac{\pi}{2} - 4\sin\frac{\pi}{4} = \pi - 4\times\frac{1}{\sqrt{2}} = \pi - 2\sqrt{2}$

The displacement of P from O when $t = \frac{\pi}{2}$ is $(\pi - 2\sqrt{2})$ m.

9 a $v = \int a\,\mathrm{d}t = \int -4\mathrm{e}^{0.2t}\,\mathrm{d}t = -20\mathrm{e}^{0.2t} + C$

When $t = 0$, $v = 20$

$20 = -20 + C \Rightarrow C = 40$

$v = 40 - 20\mathrm{e}^{0.2t}$

b $x = \int v\,\mathrm{d}t = \int (40 - 20\mathrm{e}^{0.2t})\,\mathrm{d}t = 40t - 100\mathrm{e}^{0.2t} + B$

When $t = 0$, $x = 0$

$0 = 0 - 100 + B \Rightarrow B = 100$

$x = 40t - 100\mathrm{e}^{0.2t} + 100$

The maximum value of x occurs when $\frac{\mathrm{d}x}{\mathrm{d}t} = v = 40 - 20\mathrm{e}^{0.2t} = 0$

$\Rightarrow \mathrm{e}^{0.2t} = 2$

$\Rightarrow 0.2t = \ln 2$

$\Rightarrow t = 5\ln 2$

So the maximum value of x

$= 40\times 5\ln 2 - 100\times\mathrm{e}^{0.2\times 5\ln 2} + 100 = 200\ln 2 - 100\mathrm{e}^{\ln 2} + 100$

$= 200\ln 2 - 200 + 100 = 200\ln 2 - 100$

The maximum displacement of P from O in the direction of x-increasing is $(200\ln 2 - 100)$ m.

10 a $v = \dfrac{3200}{c + dt}$

When $t = 0$, $v = 40$

$40 = \dfrac{3200}{c} \Rightarrow c = 80$

So $v = \dfrac{3200}{80 + kt} = 3200(80 + kt)^{-1}$

Hence $a = \dfrac{dv}{dt} = -3200k(80 + kt)^{-2} = -\dfrac{3200k}{(80 + kt)^2}$

When $t = 0$, $a = -0.5$

$-0.5 = -\dfrac{3200k}{80^2} \Rightarrow k = \dfrac{0.5 \times 80^2}{3200} = 1$

Solution: $c = 80$, $k = 1$

b $x = \int v \, dt = \int \dfrac{3200}{80 + t} dt = 3200 \ln(80 + t) + A$

When $t = 0$, $x = 0$

$0 = 3200 \ln 80 + A \Rightarrow A = -3200 \ln 80$

$x = 3200 \ln(80 + t) - 3200 \ln 80 = 3200 \ln\left(\dfrac{80 + t}{80}\right)$

11 a $a = \dfrac{dv}{dt} = 2e^{2t} - 11e^t + 15$

When $a = 0$

$2e^{2t} - 11e^t + 15 = 0$

$(2e^t - 5)(e^t - 3) = 0$

$e^t = 2.5, 3$

$t = \ln 2.5, \ln 3$

b $x = \int v \, dt = \int (e^{2t} - 11e^t + 15t) dt = \dfrac{e^{2t}}{2} - 11e^t + \dfrac{15t^2}{2} + C$

When $t = 0$, $x = 0$

$0 = \dfrac{1}{2} - 11 + 0 + C \Rightarrow C = \dfrac{21}{2}$

$x = \dfrac{e^{2t}}{2} - 11e^t + \dfrac{15t^2}{2} + \dfrac{21}{2}$

When $t = \ln 3$

$x = \dfrac{e^{2\ln 3}}{2} - 11e^{\ln 3} + \dfrac{15(\ln 3)^2}{2} + \dfrac{21}{2}$

$= \dfrac{9}{2} - 33 + \dfrac{15(\ln 3)^2}{2} + \dfrac{21}{2} = \dfrac{15(\ln 3)^2}{2} - 18 \approx -8.95$

As displacement is a positive quantity, the required distance is $\left(18 - \dfrac{15(\ln 3)^2}{2}\right)$ m.

12 a $a = \frac{\mathrm{d}v}{\mathrm{d}t} = 2 + \frac{1}{t+2}$

When $a = 2.2$

$$2 + \frac{1}{t+2} = 2.2 \Rightarrow t + 2 = \frac{1}{0.2} = 5 \Rightarrow t = 3$$

Note that if $a = -2.2$, which also has a magnitude of $2.2\ \mathrm{m\,s^{-2}}$, then this gives

$$2 + \frac{1}{t+2} = -2.2 \Rightarrow t + 2 = -\frac{1}{4.2} \Rightarrow t \approx -2.24$$

So this is not a valid result as $t > 0$.

b $x = \int v\,\mathrm{d}t = \int_1^4 (2t + \ln(t+2))\,\mathrm{d}t$

Using integration by parts to work out the second term of the integral, with $u = \ln(t+2)$ and $v = t$

$$\int \ln(t+2)\mathrm{d}t = \int 1 \times \ln(t+2)\mathrm{d}t$$

$$= t\ln(t+2) - \int \frac{t}{t+2}\mathrm{d}t = t\ln(t+2) - \int\left(1 - \frac{2}{t+2}\right)\mathrm{d}t$$

$$= t\ln(t+2) - t + 2\ln(t+2) = (t+2)\ln(t+2) - t$$

Hence $x = \left[t^2 + (t+2)\ln(t+2) - t\right]_1^4$

$$= (16 + 6\ln 6 - 4) - (1 + 3\ln 3 - 1)$$

$$= 12 + 6\ln 6 - 3\ln 3 = 12 + 3\ln 6^2 - 3\ln 3$$

$$= 12 + 3\ln\left(\frac{36}{3}\right) = 12 + 3\ln 12$$

So the distance moved by P in the interval $1 \leqslant t \leqslant 4$ is $(12 + 3\ln 12)$ m.

13 a $v = 3t^2 - 5t + 2$

When $v = 0$

$$3t^2 - 5t + 2 = 0$$

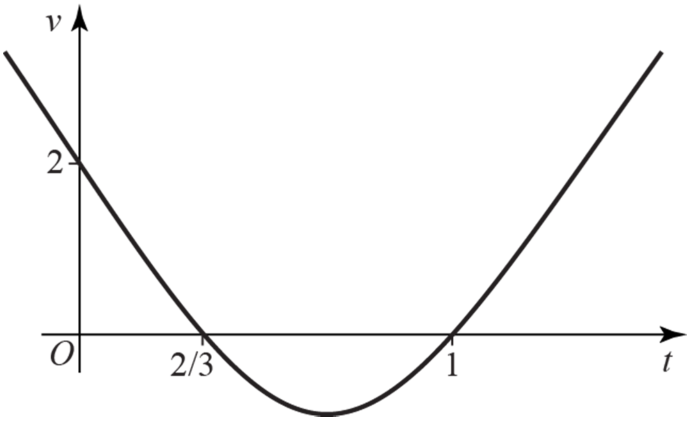
$$(3t - 2)(t - 1) = 0$$

$$t = \frac{3}{2}, t = 1$$

b $a = \frac{dv}{dt} = 6t - 5$

When $t = 5$, $a = 6 \times 5 - 5 = 25\ \mathrm{m\,s^{-2}}$

13 c The velocity-time graph for the motion of the particle is:



The particle changes direction twice in the interval $0 \leqslant t \leqslant 5$

$$\text{Total distance travelled} = \int_0^{\frac{2}{3}} (3t^2 - 5t + 2)\,\mathrm{d}t - \int_{\frac{2}{3}}^{1} (3t^2 - 5t + 2)\,\mathrm{d}t + \int_1^5 (3t^2 - 5t + 2)\,\mathrm{d}t$$

$$= \left[t^3 - \frac{5}{2}t^2 + 2t\right]_0^{\frac{2}{3}} - \left[t^3 - \frac{5}{2}t^2 + 2t\right]_{\frac{2}{3}}^{1} + \left[t^3 - \frac{5}{2}t^2 + 2t\right]_1^5$$

$$= \left(\frac{8}{27} - \frac{10}{9} + \frac{4}{3}\right) - \left(\frac{1}{2} - \frac{8}{27} + \frac{10}{9} - \frac{4}{3}\right) + \left(125 - \frac{125}{2} + 10 - \frac{1}{2}\right)$$

$$= \frac{14}{27} + \frac{1}{54} + \frac{144}{2} = \frac{28 + 1 + 3888}{54} = \frac{3917}{54} = 72.5 \text{ m (3 s.f.)}$$

d Let the displacement of P from O at time t be x m.

$$x = \int v\,dt = \int (3t^2 - 5t + 2)\,\mathrm{d}t = t^3 - \frac{5}{2}t^2 + 2t + C$$

When $t = 0$, $x = 0 \Rightarrow C = 0$

Therefore $x = t^3 - \frac{5}{2}t^2 + 2t = t\left(t^2 - \frac{5}{2}t + 2\right)$

If P returns to O, then $x = 0$ for some value of t ($t > 0$).

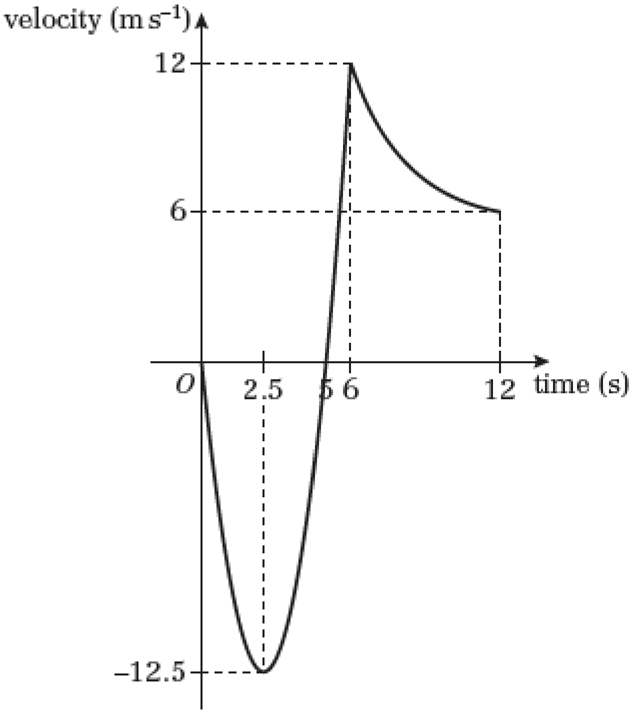
Looking for solutions of $t^2 - \frac{5}{2}t + 2 = 0$

The discriminant $(b^2 - 4ac)$ of this expression is $\left(\frac{5}{2}\right)^2 - 8 = \frac{25}{4} - \frac{32}{4} = -\frac{7}{4}$

As the discriminant is negative, this expression has no real roots. Therefore P never returns to O for any $t > 0$.

14 a For $v = 2t(t-5)$, the graph is a quadratic, with a positive x^2 coefficient, from (0, 0) to (6, 12). It cuts the x-axis at (0, 0) and (5, 0) and has a minimum at (2.5, –12.5)

For $v = \dfrac{72}{t}$, the graph is part of a reciprocal curve joining (6, 12) to (12, 6)



b For $0 \leqslant t \leqslant 6,\ a = \dfrac{dv}{dt} = 4t - 10$ This is positive when $4t < 10$, i.e. $2.5 < t \leqslant 6$

For $6 < t \leqslant 12,\ a = \dfrac{dv}{dt} = -\dfrac{72}{t^2}$ This is negative for all values of t

So the acceleration is positive for $2.5 < t \leqslant 6$

c The definite integral will be negative for the area below the x-axis in the graph in part **a**

$$\text{Total distance travelled} = -\int_0^5 (2t^2 - 10t)\,dt + \int_5^6 (2t^2 - 10t)\,dt + \int_6^{12} \frac{72}{t}\,dt$$

$$= -\left[\frac{2}{3}t^3 - 5t^2\right]_0^5 + \left[\frac{2}{3}t^3 - 5t^2\right]_5^6 + \left[72\ln t\right]_6^{12}$$

$$= \frac{125}{3} - 36 + \frac{125}{3} + 72\ln 12 - 72\ln 6$$

$$= \frac{250}{3} - \frac{108}{3} + 72(\ln 12 - \ln 6) = \left(\frac{142}{3} + 72\ln 2\right)\text{ m}$$

Challenge

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{60}{kt^2}$ for $t \geqslant 2$

So $v = \int \frac{60}{kt^2}\,\mathrm{d}t = -\frac{60}{kt} + C$

When $t = 2,\ v = 0$

$0 = -\frac{60}{2k} + C \Rightarrow C = \frac{30}{k}$

When $t = 5,\ v = 9$

$9 = -\frac{60}{5k} + C \Rightarrow C = 9 + \frac{12}{k}$

So $9 + \frac{12}{k} = \frac{30}{k} \Rightarrow 9 = \frac{18}{k} \Rightarrow k = 2$

And $C = \frac{30}{k} \Rightarrow C = 15$

So $v = 15 - \frac{30}{t}$ for $t \geqslant 2$

As $0 < \frac{30}{t} \leqslant 15$ for $t \geqslant 2, |v| < 15$

So for $t \geqslant 2$ the car never reaches a speed of 15 $\mathrm{m\,s^{-1}}$