Further centres of mass Mixed Exercise 3

1 a $V = \int \pi y^2 \mathrm{d}x = \pi \int_0^4 4x \, \mathrm{d}x$

$= \pi \left[2x^2 \right]_0^4$

$= 32\pi \text{ cm}^3$

b $M\bar{x} = \rho \int x\pi y^2 \, \mathrm{d}x = \rho\pi \int_0^4 4x^2 \mathrm{d}x$

$= \rho\pi \left[\frac{4}{3}x^3 \right]_0^4$

$= \frac{256}{3}\rho\pi$

$\therefore 32\pi\rho\bar{x} = \frac{256}{3}\pi\rho$

$\therefore \bar{x} = \frac{8}{3} \text{cm}$

2 a $V = \int \pi y^2 \mathrm{d}x = \pi \int_1^2 \frac{1}{x^2} \mathrm{d}x$

$= \pi \left[\frac{-1}{x} \right]_1^2$

$= \pi \left[\frac{-1}{2} + 1 \right]$

Volume $= \frac{\pi}{2} \text{m}^3$

b $M\bar{x} = \rho \int x\pi y^2 \mathrm{d}x = \rho\pi \int_1^2 x \times \frac{1}{x^2} \mathrm{d}x$

$= \rho\pi \int_1^2 \frac{1}{x} \mathrm{d}x$

$= \rho\pi \left[\ln x \right]_1^2$

$= \rho\pi \ln 2$

$\therefore \frac{\pi}{2}\rho\bar{x} = \rho\pi \ln 2$

$\therefore \bar{x} = 2\ln 2$

So the distance of the centre of mass from the plane face $x = 1$ is $2\ln 2 - 1 = 0.386 \text{ m}$ (3 s.f.) i.e. 39 cm to the nearest cm.

3 Let the density of the solids be ρ. Let O be the centre of the circular base of the solid.

Shape	**Mass**	**Ratio of masses**	**Distance of centre of mass from *O***
Cylinder	$\pi \times 40^2 \times 40\rho$	1	20 cm
Hemisphere	$\frac{2}{3}\pi\rho \times 40^3$	$\frac{2}{3}$	$\left(40+\frac{3}{8}\times 40\right)$cm
Solid	$\pi\rho \times 40^3\left(1+\frac{2}{3}\right)$	$\frac{5}{3}$	$\bar{x}$

$$\circlearrowleft M(O)\ \frac{5}{3}\bar{x} = 1\times 20+\frac{2}{3}\times\left(40+\frac{3}{8}\times 40\right)$$

$$= 20+\frac{110}{3}$$

$$\therefore \bar{x} = \frac{170}{5}$$

$$= 34$$

$\therefore$ The centre of mass of the solid is at a height of 34 cm above the ground.

4 Let the mass per unit volume be ρ.

Shape	**Mass**	**Mass ratios**	**Distance of centre of mass from plane face**
Cylinder	$\pi\rho r^2 \times r$	1	$\frac{r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times 2r$	$\frac{2}{3}$	$r+\frac{2r}{4}$
Model	$\pi\rho r^2 \times 1\frac{2}{3}r$	$1\frac{2}{3}$	$\bar{x}$

Note that the cylindrical base of this rocket has height r.

$$\circlearrowleft \text{M (plane face)}: 1\frac{2}{3}\bar{x} = 1\times\frac{r}{2}+\frac{2}{3}\times\left(r+\frac{2r}{4}\right)$$

$$\text{i.e. } \frac{5}{3}\bar{x} = \frac{r}{2}+\frac{2r}{3}+\frac{r}{3}$$

$$\text{i.e. } \frac{5}{3}\bar{x} = \frac{3r}{2}$$

$$\therefore \bar{x} = \frac{9r}{10}$$

$\therefore$ The centre of mass is at a distance $\frac{9r}{10}$ from the plane face.

5 a Let the density of the solid be ρ.

Shape	Mass	Mass ratios	Distance of centre of mass from C
Cylinder	$\pi\rho r^2 \times kr$	k	$-\frac{kr}{2}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$\frac{2}{3}$	$\frac{3}{8}r$
Composite body	$\pi\rho r^3\left(k+\frac{2}{3}\right)$	$k+\frac{2}{3}$	0

$$\circlearrowleft \text{M (about } C): k\times\left(-\frac{kr}{2}\right)+\frac{2}{3}\times\frac{3}{8}r=0$$

$$\therefore \frac{k^2r}{2}=\frac{r}{4}$$

$$\therefore k^2=\frac{1}{2}\Rightarrow k=\frac{1}{\sqrt{2}}=0.707 \text{ (3 s.f.)}$$

b The centre of mass of the body is at C which is always directly above the contact point.

6 a $$\bar{y}=\frac{\rho\int\frac{1}{2}y^2\,dx}{\rho\int y\,dx}=\frac{\frac{1}{2}\int_0^4\frac{x^2}{16}\left(16-8x+x^2\right)dx}{\frac{1}{4}\int_0^4 4x-x^2\,dx}$$

$$=\frac{\frac{1}{2}\int_0^4 x^2-\frac{1}{2}x^3+\frac{1}{16}x^4\,dx}{\frac{1}{4}\left[2x^2-\frac{1}{3}x^3\right]_0^4}$$

$$=2\frac{\left[\frac{1}{3}x^3-\frac{1}{8}x^4+\frac{1}{80}x^5\right]_0^4}{32-\frac{64}{3}}$$

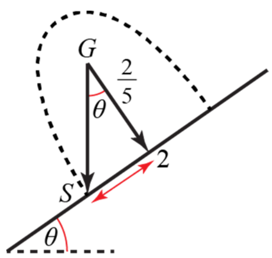
$$=\frac{6}{32}\left[\frac{64}{3}-32+\frac{64}{5}\right]$$

$$=\frac{6}{32}\times\frac{32}{15}$$

$$=\frac{6}{15}=\frac{2}{5}$$

b From symmetry the x-coordinate of the centre of mass is 2.

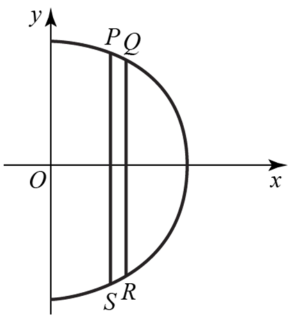
When P is about to topple the centre of mass G directly above the lower edge of the prism S.



$$\therefore \tan\theta=\frac{2}{\frac{2}{5}}=5$$

$$\therefore \theta=79° \text{ (nearest degree)}$$

7 a



Take the diameter as the y-axis and the midpoint of the diameter as the origin.

Then $M\bar{x} = \rho\int 2yx\,\mathrm{d}x$ where

$$M = \frac{1}{2}\rho\pi\left(2a^2\right) \text{and where } x^2 + y^2 = (2a)^2$$

$$\therefore 2\rho\pi\alpha^2\bar{x} = \rho\int_0^{2a} 2x\sqrt{4a^2 - x^2}\,\mathrm{d}x$$

$$= \frac{-2\rho}{3}\left[\left(4a^2 - x^2\right)^{\frac{3}{2}}\right]_0^{2a}$$

$$\therefore 2\rho\pi a^2\bar{x} = \frac{2\rho}{3}\times 8a^3$$

$$\therefore \bar{x} = \frac{16}{3}a^3 \div 2\pi a^2$$

$$= \frac{8a}{3\pi}$$

b

Shape	**Mass**	**Mass ratios**	**Centre of mass (distance from AB)**
Large semicircle	$2\pi\rho a^2$	4	$\frac{8a}{3\pi}$
Semicircle diameter AD	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Semicircle diameter OB	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Remainder	$\pi\rho a^2$	2	$\bar{x}$

$$\circlearrowleft MO: 4\times\frac{8a}{3\pi} - 1\times\frac{4a}{3\pi} - 1\times\frac{4a}{3\pi} = 2\bar{x}$$

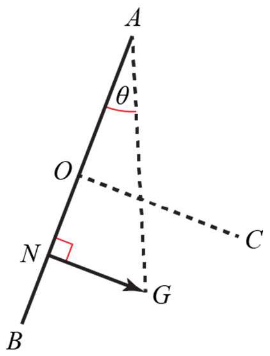
$$\therefore \frac{24a}{3\pi} = 2\bar{x}$$

$$\therefore \bar{x} = \frac{4a}{\pi}$$

c The distance from OC is a

The distance from OB is $\frac{2a}{\pi}$

7 d



Let N be the foot of the perpendicular from G onto AB.
In the diagram θ is the angle between AB and the vertical.
From ΔANG

$$\tan\theta = \frac{NG}{AN} = \frac{\frac{2a}{\pi}}{2a+a}$$

$$= \frac{2}{3\pi}$$

$\therefore \theta = 12°$ (to the nearest degree)

$\therefore$ The angle between AB and the horizontal is $90-12=78°$ (to the nearest degree)

8 a

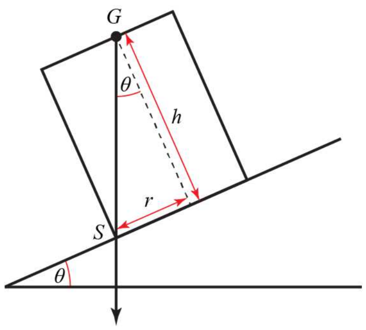
Shape	Mass	Mass ratios	Distance of centre of mass from *O*
Cylinder	$\pi\rho r^2 h$	h	$-\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi\rho(3r)^3$	$18r$	$\frac{3}{8}(3r)$
Mushroom	$\pi\rho r^2(h+18r)$	$h+18r$	0

$$\circlearrowleft M(O): -h\times\frac{h}{2}+18r\times\frac{3}{8}\times 3r = 0$$

$$\therefore \frac{h^2}{2} = \frac{81r^2}{4}$$

$$\therefore h = r\sqrt{\frac{81}{2}}$$

b



When the mushroom is about to topple GS is vertical.

From the diagram $\tan\theta = \frac{r}{h}$

$$= \sqrt{\frac{2}{81}}$$

$\therefore \theta = 9°$ (nearest degree)

9 a $V = \pi\int y^2\,dx$

$$= \pi\int_0^a 4ax\,dx$$

$$= \pi\left[2ax^2\right]_0^a$$

$$= 2\pi a^3$$

9 b $\bar{x} = \dfrac{\pi\int xy^2\,dx}{\pi\int y^2\,dx}$

$= \dfrac{\pi\int_0^a 4ax^2\,dx}{2\pi a^3}$

$= \pi\dfrac{\left[\dfrac{4ax^3}{3}\right]_0^a}{2\pi a^3}$

$= \dfrac{\frac{4}{3}\pi ax^4}{2\pi a^3}$

$= \dfrac{2}{3}a$

c

Shape	Mass	Mass ratios	Distance of centre of mass from X
S_1	$2\pi\rho a^3$	ρ_1	$-\dfrac{a}{3}$
S_2	$\dfrac{2}{3}\pi\rho_2(2a)^3$	$\dfrac{8}{3}\rho_2$	$\dfrac{3}{8}(2a)$
Combined solid	$2\pi a^3(\rho_1+\dfrac{8}{3}\rho_2)$	$\rho_1+\dfrac{8}{3}\rho_2$	0

X is the centre of the common plane base.

$\circlearrowleft \mathrm{M}(\mathrm{X})$:

$-\rho_1\times\dfrac{a}{3}+\dfrac{8}{3}\rho_2\times\dfrac{6a}{8}=0$

$\therefore \dfrac{1}{3}\rho_1 = 2\rho_2$

$\therefore \rho_1 = 6\rho_2$

$\rho_1 : \rho_2 = 6:1$

d Given that $\rho_1 : \rho_2 = 6:1$, then as centre of mass is at centre of hemisphere this will always be above the point of contact with the plane when a point of the curved surface area of the hemisphere is in contact with a horizontal plane.
(Tangent – radius property)

10 a

Shape	Mass	Mass ratios	Distance of centre of mass from *AB*
Cylinder	$\pi\rho(2r)^2 \times 3r$	$12r$	$\frac{3r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times h$	$\frac{1}{3}h$	$\frac{1}{4}h$
Remainder	$\pi\rho(12r^3 - \frac{1}{3}r^3h)$	$12r - \frac{1}{3}h$	$\bar{x}$

$$\circlearrowleft: \left(12r - \frac{1}{3}h\right)\bar{x} = 12r \times \frac{3r}{2} - \frac{1}{3}h \times \frac{1}{4}h$$

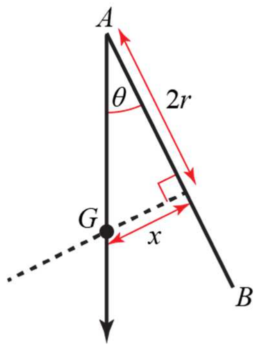
$$\therefore \left(12r - \frac{1}{3}h\right)\bar{x} = 18r^2 - \frac{1}{12}h^2$$

$$\therefore \bar{x} = \frac{18r^2 - \frac{1}{12}h^2}{12r - \frac{1}{3}h}$$

Multiply numerator and denominator by 12

$$\therefore \bar{x} = \frac{216r^2 - h^2}{4(36r - h)}$$

b



From the diagram

$$\tan\theta = \frac{\bar{x}}{2r}$$

$$\text{As } h = 2r, \bar{x} = \frac{216r^2 - (2r)^2}{4(36r - 2r)} = \frac{212r^2}{136r} = \frac{53}{34}r$$

$$\therefore \tan\theta = \frac{53}{68}$$

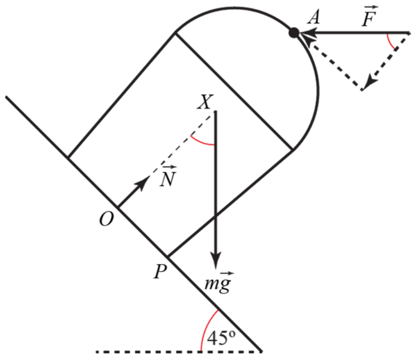
$$\therefore \quad \theta = 38° \ \text{(nearest degree)}$$

11 a First find the centre of mass of the frustum. The centre of mass of the full cone is $\frac{1}{4}h$ from its base, on the symmetry axis. Here h is the height of the full cone, which can be found using similar triangles $\frac{10}{h} = \frac{5}{h-30} \Rightarrow h = 60$ cm. Now taking moments about the base of the cone

$$\frac{1}{3}\pi 10^2 h \times \frac{1}{4}h - \frac{1}{3}\pi 5^2(h-30) \times \left(\frac{1}{4}h - \frac{1}{4}30 + 30\right)$$

$= \frac{1}{3}\pi\left(10^2 h - 5^2(h-30)\right)\bar{x} \ \Rightarrow \bar{x} = \frac{165}{14}$ cm. Now take moments about the centre of the common plane of the frustum and the solid hemisphere $\frac{1}{3}\pi\rho\left(10^2 \times 60 - 5^2 \times 30\right)\bar{x} - 3\rho\frac{2}{3}\pi 10^3 \times \frac{3}{8} \times 10 =$ $\left(\frac{1}{3}\pi\rho\left(10^2 \times 60 - 5^2 \times 30\right) + 3\rho\frac{2}{3}\pi 10^3\right)\bar{X} \ \Rightarrow \bar{X} = 3.5$ cm below their common plane. Thus, the centre of mass of the compound solid is 26.5 cm from its base.

11 b We want to take moments about the lowest base point P, using the fact that distance OP is 5 cm, OA is 40 cm and OX is 26.5 cm.



There are three forces acting on the body, namely $\vec{F}$, $\vec{N}$ and $\overrightarrow{mg}$. At the point of toppling, the reaction force is acting through the point P. It is easiest to decompose the forces into directions parallel and normal to the plane $F\sin 45^\circ \times 40 + F\cos 45^\circ \times 5 + mg\cos 45^\circ \times 5 = mg\sin 45^\circ \times 26.5$

$$\frac{mg}{\sqrt{2}}\times 5 + \frac{F}{\sqrt{2}}\times 5 + \frac{F}{\sqrt{2}}\times 40 = \frac{mg}{\sqrt{2}}\times 26.5 \Rightarrow F = \frac{26.5-5}{40+5}mg \approx 0.478mg \text{ (3 s.f.)}$$

12 a The mass is $M = \rho\int_2^4 \pi y^2\,\mathrm{d}x = \rho\int_2^4 \pi\left(\frac{2}{x+3}\right)^2\mathrm{d}x$

$$= 4\pi\rho\left[-\frac{1}{x+3}\right]_2^4 = \frac{8}{35}\pi\rho$$

The centre of mass $M\bar{x} = \rho\int_2^4 \pi x y^2\,\mathrm{d}x = 4\rho\int_2^4 \pi\frac{x}{(x+3)^2}\mathrm{d}x$

$$= 4\rho\pi\int_2^4 \frac{x}{(x+3)^2}\mathrm{d}x$$

$$= 4\rho\pi\int_2^4\left(\frac{1}{x+3} - \frac{3}{(x+3)^2}\right)\mathrm{d}x$$

$$= 4\rho\pi\left(\left[\ln(x+3)\right]_2^4 - 3\int_2^4 \frac{1}{(x+3)^2}\mathrm{d}x\right)$$

$$= 4\rho\pi\left[\ln(x+3) + \frac{3}{x+3}\right]_2^4$$

$$= 4\rho\pi\left(-\frac{6}{35} + \ln\frac{7}{5}\right) \Rightarrow \bar{x} \approx 2.89.$$

Thus, the centre of mass of the solid above the ground is 1.11.

b The radius of the smaller circular end is $y(x=4) = \frac{2}{7}$

The angle at the point of tipping is $\tan\theta = \frac{\frac{2}{7}}{4-\bar{x}} \approx 0.2570 \Rightarrow \theta = 14.4^\circ$ (3 s.f.).

13 a The mass is given by $M = \int_0^9 m(x)\,dx = \int_0^9 1000 + 400\sqrt{x}\,dx$

$$= \left[1000x + \frac{800}{3}x^{3/2}\right]_0^9 = 16\,200 \text{ kg.}$$

b Using the formula $M\bar{x} = \int_0^9 xm(x)\,dx = \int_0^9 1000x + 400x\sqrt{x}\,dx$

$$= \left[500x^2 + 160x^{5/2}\right]_0^9 = 79\,380 \Rightarrow$$

$\bar{x} = 4.9$ m.

14 a The mass is given by $M = \int_0^{30}\left(20 - \frac{1}{9}h\right)dh = \left[20h - \frac{1}{18}h^2\right]_0^{30}$

$= 1350$ g.

The centre of mass $M\bar{x} = \int_0^{30} h\left(20 - \frac{1}{2}h\right)dh = \left[20h^2 - \frac{1}{4}h^3\right]_0^{30}$

$= 54\,000 \Rightarrow \bar{x} = 40$ cm.

b i *e.g.* Suitable if uniform across cross-section, or suitable as height $\gg$ diameter, or unsuitable as may be non-uniform across cross-section.

ii Unsuitable as rod has no width so will never be stable.

c If the body is about to topple, $\tan\theta = \frac{4}{\bar{x}} = 0.1 \Rightarrow \theta \approx 5.71^\circ$ (3 s.f.).

15 The mass of the rod is

$$M = \int_0^{12} 2 + \frac{1}{4}x^2\,dx$$

$$= \left[2x + \frac{1}{12}x^3\right]_0^{12} = 168 \text{ kg.}$$

The centre of mass $M\bar{x} = \int_0^{12} 2x + \frac{1}{4}x^3\,dx$

$$= \left[x^2 + \frac{1}{16}x^4\right]_0^{12} = 1440 \Rightarrow$$

$\bar{x} = \frac{60}{7} \approx 8.5714$ m from point A.

Resolving vertically $T_A + T_B = Mg$ and taking moments about point A, $Mg\bar{x} = T_B \times 12$.

Solving these equations gives

$$T_B = \frac{1}{12}Mg\bar{x} = \frac{1}{12}168 \times \frac{60}{7} \times g = 120g \text{ N,}$$

$$T_A = \frac{1}{12}Mg(12 - \bar{x})$$

$$= \frac{1}{12}168 \times \left(12 - \frac{60}{7}\right)g = 48g \text{ N.}$$

16 The centre of mass of the rod

$$\bar{x} = \frac{\int_0^{4l} x(35-\frac{1}{2}x)\,dx}{\int_0^{4l} 35-\frac{1}{2}x\,dx} = \frac{\left[17.5x^2-\frac{1}{6}x^3\right]_0^{4l}}{\left[35x-0.25x^2\right]_0^{4l}}$$

$$= \frac{2l(105-4l)}{3(35-l)}$$

Resolving forces vertically

$2T = Mg = 4l(35-l)g$. Taking moments about point A $Mg\bar{x} = T\times 3l$. This gives

$T = 600g$ N and $l = 15$ m. Thus the length of the rod is $4\times 15 = 60$ m.

Challenge

a $M = \int_0^h \pi(h-x)^2(x+1)\,dx$

$$= \pi\int_0^h h^2 - 2hx + h^2x + x^2 - 2hx^2 + x^3\,dx$$

$$= \pi\left[\frac{1}{12}x\left(6h^2(2+x) - 4hx(3+2x) + x^2(4+3x)\right)\right]_0^h$$

$$= \frac{1}{12}\pi h^3(4+h)$$

b $M\,\bar{x} = \int_0^h \pi x(h-x)^2(x+1)\,dx$

$$= \pi\int_0^h h^2x + (-2+h)hx^2 + (1-2h)x^3 + x^4\,dx$$

$$= \pi\left[\frac{1}{60}x^2\left(10h^2(3+2x) - 10hx(4+3x) + 3x^2(5+4x)\right)\right]_0^h$$

$$= \frac{1}{60}\pi h^4(5+2h) \Rightarrow$$

$$\bar{x} = \frac{h(5+2h)}{5(4+h)} = \frac{1}{3}h \Rightarrow h = 5\text{ m}.$$

c Suppose $\bar{x} = \dfrac{h(5+2h)}{5(4+h)} = kh$ for some constant k.

Then $k = \dfrac{5+2h}{5(4+h)} = \dfrac{2}{5} - \dfrac{3}{5(h+4)}$, as $h\to\infty$, $k\to\dfrac{2}{5}$.

Hence as h varies the height of the centre of mass of the cone above its base cannot exceed $\dfrac{2}{5}h$.