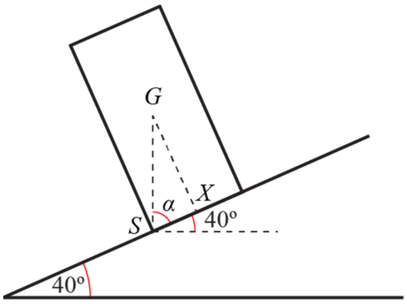
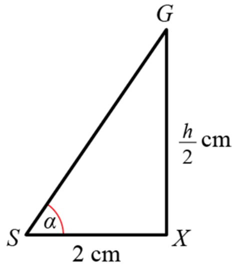
Further centres of mass 3E

1





Let the maximum height be h cm. The cylinder is about to topple and so its centre of mass G is directly above the point S on the circumference of the base. X is the midpoint of the base.

As $\alpha + 40° = 90°, \alpha = 50°$.

In $\Delta GSX, SX = 2\text{ cm}$ (radius)

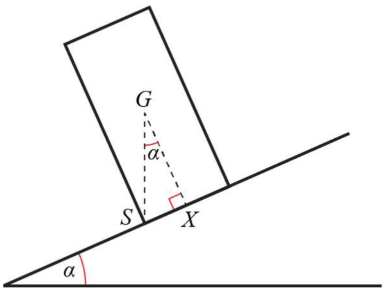
$$GX = \frac{h}{1}(\text{position of centre of mass})$$

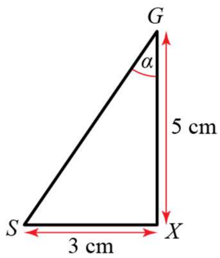
$$\therefore \tan 50° = \frac{\frac{h}{2}}{2}$$

$$\therefore h = 4\tan 50°$$

$$\therefore h = 4.77\text{ cm}\,(3\text{ s.f.})$$

2





a When the cylinder is about to topple, G is vertically above point S. X is the midpoint of the base.

Let α be the angle which the plane makes with the horizontal.

In triangle $GSX, S\hat{G}X = \alpha$

$$GX = \frac{1}{2}\times 10\text{ cm} = 5\text{ cm}\,(\text{position of centre of mass})$$

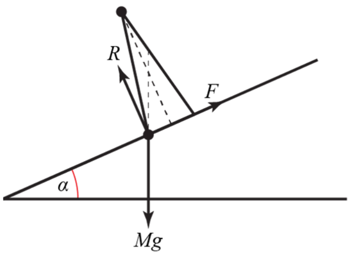
$$SX = 3\text{ cm} \quad (\text{radius})$$

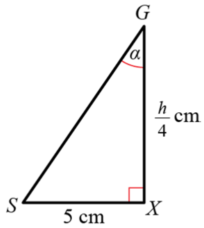
$$\therefore \tan\alpha = \frac{3}{5}$$

i.e. $\alpha = 31°$ (to the nearest degree)

b The equilibrium is maintained if $\tan\theta = \frac{3}{5}$. At the point of slipping, $F = \mu R$, where F is frictional force, R reactive force and μ is the coefficient of friction. Resolving forces in the direction orthogonal to the plane $R - Mg\cos\theta = 0$, and parallel to the plane $F - Mg\sin\theta = 0$. These two conditions imply that $\mu = \tan\theta$ at the point of slipping. Hence $\mu = \frac{3}{5}$.

3





a When the cone is about to slide $F = \mu R$

$$\text{i.e.}\ F = \frac{\sqrt{3}}{3}R \qquad \textbf{(1)}$$

R(↗)

Then $F - mg\sin\alpha = 0 \quad \therefore F = Mg\sin\alpha$ **(2)**

R(↖)

Then $R - Mg\cos\alpha = 0 \quad \therefore R = Mg\cos\alpha$ **(3)**

Substituting F and R into equation **(1)**

Then $Mg\sin\alpha = \frac{\sqrt{3}}{3}Mg\cos\alpha$

$\therefore \tan\alpha = \frac{\sqrt{3}}{3}$

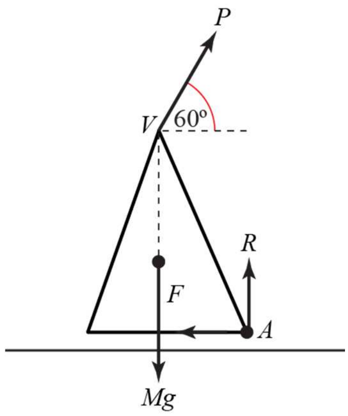
$\therefore \quad \alpha = 30°$

b From ΔGSX, where G is the centre of mass of the cone, X the centre of its base and S a point on the circumference of the base about which topping is about to occur:

$$\tan\alpha = \frac{5}{\frac{h}{4}} = \frac{20}{h}$$

$$\therefore \quad h = \frac{20}{\tan\alpha} = 20 \div \frac{\sqrt{3}}{3} = 20\sqrt{3} = 35\,\text{cm}\,(2\text{ s.f.})$$

4 a



Let the point about which toppling occurs be A.

Take moments about point A.

When toppling is about to occur, R and F act through point A.

So $P\cos 60 \times 2r + P\sin 60 \times r = Mg \times r$

$$\therefore Pr + \frac{P\sqrt{3}}{2}r = Mgr$$

$$\therefore P\left(1 + \frac{\sqrt{3}}{2}\right) = Mg$$

So $P = \dfrac{2Mg}{2+\sqrt{3}}$

b R($\rightarrow$)

$P\cos 60° - F = 0$

$\therefore \quad F = \dfrac{Mg}{2+\sqrt{3}}$

$R(\uparrow)$

$P\sin 60° + R - Mg = 0$

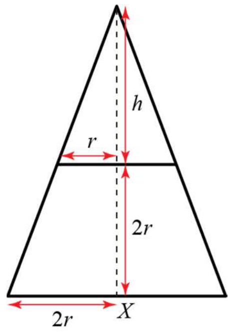
$\therefore R = Mg - \dfrac{Mg\sqrt{3}}{2+\sqrt{3}} = \dfrac{2Mg}{2+\sqrt{3}}$

As the cone is on the point of slipping, $F = \mu R$

$\therefore \mu = F \div R = \dfrac{1}{2}$

i.e. μ, the coefficient of friction, $= \dfrac{1}{2}$

5 a



Let the height of the small cone shown be h.

Using similar triangles

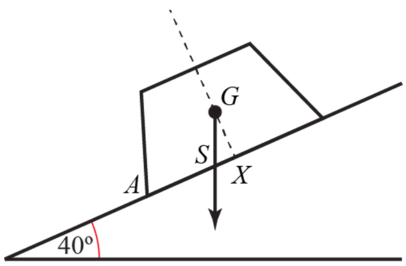
$$\frac{h}{h+2r} = \frac{r}{2r}$$
$$\therefore 2h = h + 2r$$
$$\therefore h = 2r$$

Shape	Mass	Ratio of masses	Distance of centre of mass from X
Large cone	$\rho\frac{1}{3}\pi(2r)^2(4r)$	8	r
Small cone	$\rho\frac{1}{3}\pi r^2 \times 2r$	1	$2r + \frac{2r}{4} = \frac{5r}{2}$
Frustum	$\rho\frac{1}{3}\pi \times 14r^3$	7	$\bar{x}$

Take moment about X:

$$8r - \frac{5r}{2} = 7\bar{x}$$
$$\therefore \bar{x} = \frac{11r}{14}$$

b i

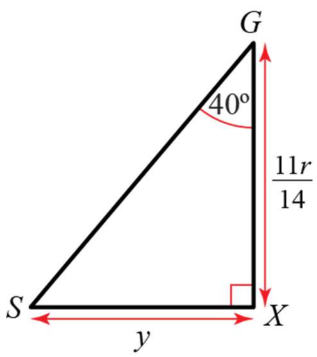


Let G be the position of the centre of mass.

Let S be the point an the plane vertically below G.

Let X be the centre of the circular face with radius $2r$ and A be the point about which tilting would occur.

If $SX < AX$ then the solid rests in equilibrium without toppling



Let $SX = y$.

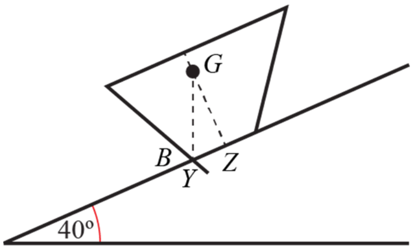
As $SX = 0.66r$ and $AX = 2r$

$SX < AX$ and the solid rests without toppling

Then $\tan 40° = \frac{y}{\frac{11r}{14}}$

$$\therefore y = \frac{11r}{14}\tan 40° = 0.66r \text{ (2 s.f.)}$$

5 b ii



This time Y is vertically below G. Z is the centre of the circular face and B is the point about which toppling would occur.

If $YZ > BZ$ then toppling occurs.

As $YZ = 1.02r$ and $BZ = r$

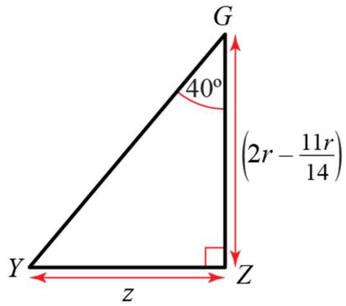
$YZ > BZ$ and toppling would occur.

Let $YZ = z$

$$\text{Then } \tan 40° = \frac{z}{\frac{17r}{14}}$$

$$\therefore z = \frac{17r}{14}\tan 40° = 1.02r$$

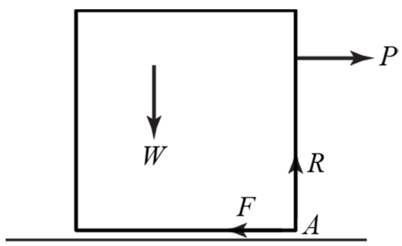
c



As the angle of slope is 40° limiting friction would imply $\mu = \tan 40°$.

No slipping implies $\mu \geqslant 0.839$ (3 s.f.)

6



a Consider the cube in equilibrium, on the point of toppling, so R acts through the corner A.

$\text{R}(\rightarrow): P - F = 0 \therefore F = P$

$\text{R}(\uparrow): R - W = 0 \therefore R = W$

$\circlearrowleft M(A): P \times 4a = W \times 3a$

$$\therefore P = \frac{3}{4}W$$

If equilibrium is broken by toppling $P = \frac{3}{4}W$, so $F = \frac{3}{4}W$

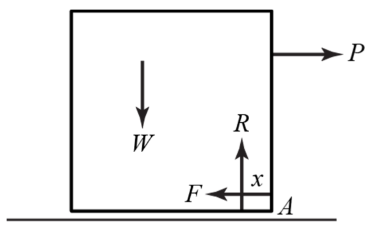
But $F < \mu R$

$\therefore \frac{3}{4}W < \mu W$ so $\mu > \frac{3}{4}$ is the condition for toppling.

If however, $\mu < \frac{3}{4}$ then the cube will be on the point of slipping when $F = \mu R$

i.e. when $P = \mu W$ the cube will start to slip.

b Let R act at a point x from A.



$\text{R}(\rightarrow) P - F = 0 \therefore P = F$

$\text{R}(\uparrow) R - W = 0 \therefore R = W$

When the cube is about to slip: $F = \mu R$

$$\therefore P = \frac{1}{4}W$$

$\circlearrowleft M(A)$: $P \times 4a + Rx = W \times 3a$ (substitute for P)

$$\therefore \frac{1}{4}W \times 4a + Rx = W \times 3a \text{ (substitute for } R\text{)}$$

$$\therefore Wx = W \times 2a$$

$$\text{i.e. } x = 2a$$

The required distance is $2a$.

7 **a** We first find the centre of mass of the composite body. Taking moments about the bottom face of body B $M_A(10+30)+M_B 15=(M_A+M_B)\bar{y} \Rightarrow \bar{y}=\frac{5(8M_A+3M_B)}{M_A+M_B}=\frac{5(8+3k)}{1+k}$

Taking moments about their common face $M_A 15+M_B 10=(M_A+M_B)\bar{x} \Rightarrow$

$\bar{x}=\frac{5(3M_A+2M_B)}{M_A+M_B}=\frac{5(3+2k)}{1+k}$

Given that $k=5$, $M_A=20\times20\times30\times\rho=12\times10^3\rho$ and $M_B=20\times20\times30\times5\rho=60\times10^3\rho$,

$\bar{y}=\frac{115}{6}\approx19.2\,\text{cm}$, $\bar{x}=\frac{65}{6}\approx10.8\,\text{cm}$. The angle is $\tan\theta=\frac{20-\bar{x}}{\bar{y}}=\frac{1+2k}{8+3k}$

$=\frac{11}{23}\approx0.478 \Rightarrow \theta\approx25.6^\circ$ (3 s.f.).

b We have that $\tan\theta=\frac{1+2k}{8+3k}$

For the cuboid A to topple, $\tan\theta=\frac{20-15}{10}=\frac{1}{2}$

Solving $\frac{1+2k}{8+3k}<\frac{1}{2} \Rightarrow 0<k<6$

8 Slicing the cylinder into thin horizontal slices we can integrate the mass as $M=\int_0^l \pi r^2\rho(x)\,\mathrm{d}x$, where $l=1.5$ m is the length of the cylinder, $r=0.25$ m is the radius, and $\rho(x)=\cosh x$ kg m^{-3} is its mass density. Integrating

$M=\int_0^{1.5}\pi0.25^2\cosh x\,\mathrm{d}x=\pi0.25^2[\sinh x]_0^{1.5}$

$=\pi0.25^2\sinh1.5\approx0.418$.

The centre of mass

$M\bar{x}=\int_0^l \pi r^2 x\rho(x)\,\mathrm{d}x=\int_0^{1.5}\pi0.25^2 x\cosh x\,\mathrm{d}x$

$=\pi0.25^2\left([x\sinh x]_0^{1.5}-\int_0^{1.5}\sinh x\,\mathrm{d}x\right)$

$=\pi0.25^2[x\sinh x-\cosh x]_0^{1.5}\simeq0.3616$

$\Rightarrow \bar{x}\approx0.865$ m (3 s.f.). The maximum angle is given by $\tan\theta=\frac{r}{\bar{x}}\approx16.1^\circ$.

9 a

Shape	Mass	Mass ratios	Distance of centre of mass from *O*
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$h+\frac{3}{8}r$
Cylinder	$\pi\rho r^2 h$	$3h$	$\frac{h}{2}$
Composite solid	$\pi\rho r^2\left(\frac{2}{3}r+h\right)$	$2r+3h$	$\bar{x}$

$$\circlearrowleft M: 2r\left(h+\frac{3}{8}r\right)+3h\times\frac{h}{2}=(2r+3h)\bar{x}$$

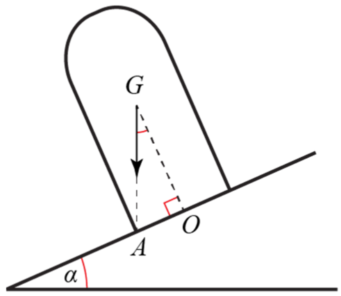
$$\therefore 2rh+\frac{3}{4}r^2+\frac{3}{2}h^2=(2r+3h)\bar{x}$$

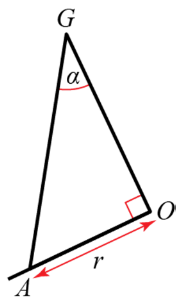
Multiply both sides by 4

$$8rh+3r^2+6h^2=4(2r+3h)\bar{x}$$

$$\therefore \bar{x}=\frac{6h^2+8hr+3r^2}{4(3h+2r)}$$

b When the solid is on the point of toppling the centre of mass G is vertically above point A as shown.





In ΔGOA,

$\angle AGO=\alpha$

$OA=r$

and $OG=\frac{6(3r)^2+8(3r^2)+3r^2}{4(9r+2r)}$

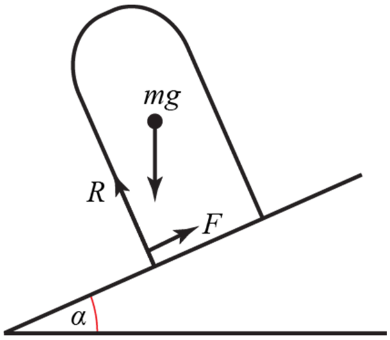
(i.e. $\bar{x}$ with $h=3r$)

$$\therefore OG=\frac{81r^2}{44r}=\frac{81r}{44}$$

$$\therefore \tan\alpha=\frac{r}{\frac{81}{44}r}=\frac{44}{81}$$

$\therefore \quad \alpha=29°$ (nearest degree)

9 c



$R(\nearrow)\ F - mg\sin\alpha = 0 \therefore F = mg\sin\alpha$

$R(\nwarrow)\ R - mg\cos\alpha = 0 \therefore R = mg\cos\alpha$

The solid does not slip

$\therefore F \leqslant \mu R$

i.e., $mg\sin\alpha \leqslant \mu mg\cos\alpha$

$\therefore \mu \geqslant \tan\alpha$

i.e.: $\mu > \frac{44}{81}$ if the solid did not slip before it toppled.

[if $\mu = \frac{44}{81}$ it slips and topples at the same time.]

10 a Let the mass per unit volume be ρ.

Shape	Mass	Mass ratio	Position of centre of mass – distance from *O*
Large cone	$\frac{1}{3}\pi\rho(2r)^2 2h$	8	$\frac{2h}{4}$
Small cone	$\frac{1}{3}\pi\rho r^2 h$	1	$h+\frac{h}{4}$
Frustum	$\frac{1}{3}\pi\rho\left(8r^2h - r^2h\right)$	7	$\bar{x}$

The centre of the base is the point O.

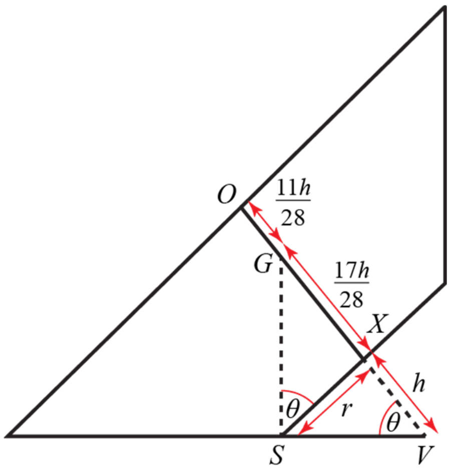
The radius of the small cone is obtained by singular triangles.

$$\circlearrowleft MO: 8\times\frac{2h}{4} - 1\times\frac{5h}{4} = 7\bar{x}$$

$$\therefore \frac{11h}{4} = 7\bar{x}$$

$$\text{i.e. } \bar{x} = \frac{11}{28}h$$

10 b As $OG = \frac{11h}{28}, GX = h - \frac{11h}{28}$
$= \frac{17h}{28}$



From $\Delta S\,GXS$ and VXS shown:

$\tan\theta = \frac{\frac{17h}{28}}{r}$ and $\tan\theta = \frac{r}{h}$

Eliminating r, $h\tan\theta = \frac{\frac{17h}{28}}{\tan\theta}$

$h\tan\theta = \frac{\frac{17h}{28}}{\tan\theta}$

$\therefore \tan^2\theta = \frac{17}{28}$

$\therefore \theta = 38°$(nearest degree)

11 a At the point of sliding, the frictional force $F = \mu R$, where R is the reaction force. Resolving forces in the direction normal to the plane $R = Mg$, and horizontally $P = F$. From this we can find that $P > \mu Mg$.

b Taking moments about the point of contact with the plane, when the cone is just about to tilt $Mgr = Ph \Rightarrow P > \frac{r}{h}Mg = \frac{3}{8}Mg$.

c i The force required for the cone to tilt is greater than the one for it to slide. For $\mu = \frac{1}{4}$ the cone will slide.

ii For $\mu = \frac{1}{2}$ the cone will tilt.

iii For $\mu = \frac{3}{8}$ the cone will remain stationary, perfectly balanced between the opposing forces, as the force $P = \frac{3}{8}Mg$ is not sufficient to cause the cone to either tilt or topple, both of which require $P > \frac{3}{8}Mg$.

12 a Let the mass of the cylinder be M, height h, and radius r. Suppose the cylinder is about to topple. Taking moments about the highest point of the base O, $Ph\cos\alpha = Mgx$, where x is the shortest distance between the force Mg and the point O. We can find it using similar triangles

$\tan\alpha = \frac{y}{h/2} = \frac{\sqrt{(y+r)^2 - x^2}}{x} \Rightarrow y = \frac{1}{2}h\tan\alpha = \frac{1}{2}\times 4\times\frac{3}{4} = 1.5$ cm, and $x = 3.6$ cm. Also note that $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$. Thus $P = (\cos\alpha)^{-1}\frac{x}{h}Mg = \frac{5}{4}\times\frac{3.6}{4}\times 0.2\times g = \frac{9}{40}g$ N.

12 b Resolving forces horizontally to the plane (at the point of slipping) $P\cos\alpha = Mg\sin\alpha + \mu R$, and vertically $R - P\sin\alpha = Mg\cos\alpha$. Hence $P = \frac{gM(3+4\mu)}{4-3\mu} = \frac{3}{10}g$ N.

c As $\frac{3}{10}g > \frac{9}{40}g$ the cylinder topples before it slides.

Challenge

a Let the mass per unit volume of the solids be ρ. Let O be the centre of the plane circular faces which coincide.

Shape	Mass	Ratio of masses	Distance of centre of mass from O
Cone	$\frac{1}{3}\pi\rho r^2 h$	h	$\frac{h}{4}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$\frac{-3r}{8}$
Toy	$\frac{1}{3}\pi\rho\left(r^2h + 2r^3\right)$	$h + 2r$	$\bar{x}$

$$\circlearrowleft O\ (h+2r)\bar{x} = h \times \frac{h}{4} + 2r\left(\frac{-3r}{8}\right)$$
$$= \frac{h^2}{4} - \frac{3r^2}{4}$$
$$\therefore \bar{x} = \frac{\left(h^2 - 3r^2\right)}{4(h+2r)}$$

b i If $h > r\sqrt{3}$ then $\bar{x} > 0$ so the centre of mass is in the cone – the cone will fall over.

ii If $h < r\sqrt{3}$ then $\bar{x} < 0$ so the centre of mass is in the hemisphere, the toy will return to vertical position.

iii If $h = r\sqrt{3}$, then $\bar{x} = 0$ so the centre of mass is on the join at point O. The toy will remain in equilibrium in its new position.