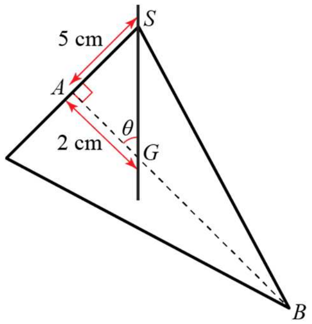
Further centres of mass 3D

1



The diagram shows the equilibrium position with the centre of mass G, vertically below the point of suspension S.

As $AG = \frac{1}{4}h$ for a cone

$\therefore AG = 2\text{cm}$

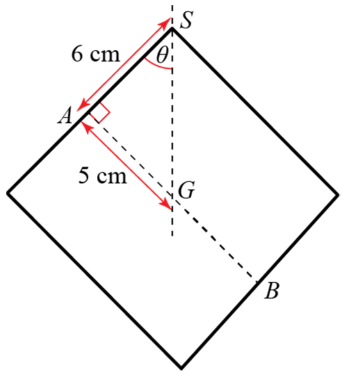
Also the radius $AS = 5\text{cm}$.

Let the angle between the vertical and the axis be θ .

Then from $\Delta ASG, \tan\theta = \frac{5}{2}$

$\therefore \theta = 68^\circ$ (to the nearest degree)

2



The diagram shows the equilibrium position with the centre of mass G below the point of suspension S.

As $AG = \frac{1}{2}h$ for a uniform cylinder

$\therefore AG = 5\text{cm}$

Also the radius $AS = 6\text{cm}$.

The angle between the vertical and the circular base of the cylinder is θ.

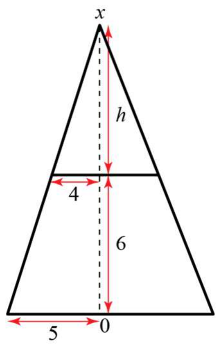
From $\Delta ASG, \tan\theta = \frac{5}{6}$

$\therefore \theta = 40^\circ$ **(to the nearest degree)**

3 The distance from the centre of mass to the base is $\frac{1}{2}r$ from the centre. The angle between the axis of the shell and the downward vertical when the shell is in equilibrium

$\tan\theta = \frac{r}{\frac{1}{2}r} = 2 \Rightarrow \theta = \arctan 2 \approx 63.4^\circ$ (3 s.f.).

4 a



From similar triangles

$$\frac{h}{h+6} = \frac{4}{5}$$

$$\therefore 5h = 4h + 24$$

i.e. $h = 24$

Centre of mass lies at the axis of symmetry OX.

Shape	Mass	Mass ratios	Position of centre of mass i.e. distance from *O*
Large cone	$\frac{1}{3}\pi\rho \times 5^2 \times 30$	125	$\frac{30}{4} = 7.5$
Small cone	$\frac{1}{3}\pi\rho \times 4^2 \times 24$	64	$6 + \frac{24}{4} = 12$
Frustum	$\frac{250\pi}{3}\rho - 128\pi\rho$	61	$\bar{x}$

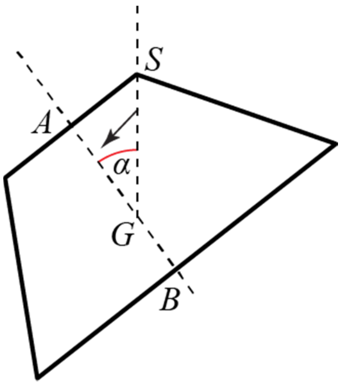
Take moments about O

$$125 \times 7.5 - 64 \times 12 = 61\bar{x}$$

$$\therefore 169.5 = 61\bar{x}$$

$$\therefore \bar{x} = 2.78 \text{ (3 s.f.)} \left(\text{or } \frac{339}{122} \right)$$

b



In equilibrium the centre of mass G lies vertically below the point of suspension S.

Let the required angle be α.

AS is smaller radius = 4 cm

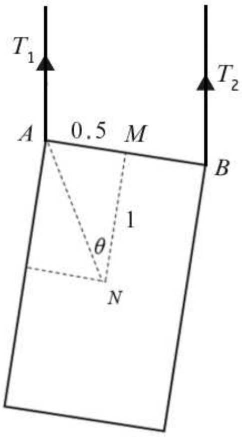
$AG = 6 - 2.78 = 3.22\,\text{cm (3 s.f.)}$

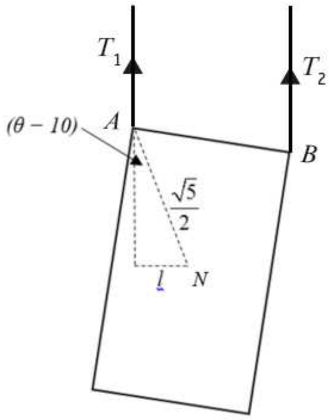
$$\tan\alpha = \frac{AS}{AG} = \frac{4}{3.22}$$

$\therefore \alpha = 51°$ (to the nearest degree)

5 $\tan\theta = \frac{\frac{1}{2}}{1} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.565...$

$AN = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$





$\sin(\theta - 10) = \frac{l}{\sqrt{\frac{5}{2}}} \Rightarrow l = 0.3187...$

perpendicular distance, x, of B from A is

$\cos 10 = \frac{x}{1} \Rightarrow x = 0.9848...$

Taking moments about A

$0.3187 \times 2g = 0.9848T_2$

$T_2 = 6.34$ N (3 s.f.)

Since $T_1 + T_2 = 2g$

$T_1 = 13.3$ N (3 s.f.)

6 a The mass density of the rod is given by $\rho = \dfrac{10}{\sqrt{1+h}}$ kg/m. The mass of the rod

$M = \int_0^1 \rho\, dh = \int_0^1 \dfrac{10}{\sqrt{1+h}} dh = 20\left[\sqrt{1+h}\right]_0^1 = 20\left(\sqrt{2}-1\right) \approx 8.28$ kg. The centre of mass of the rod is

given by $M\bar{x} = \int_0^1 h\rho\, dh = \int_0^1 h\dfrac{10}{\sqrt{1+h}} dh$. To do this integral make a substitution $u = h+1$

$\int \dfrac{h}{\sqrt{1+h}} dh = \int \dfrac{u-1}{\sqrt{u}} du = \int \sqrt{u} - \dfrac{1}{\sqrt{u}} du$

$= \frac{2}{3}u^{3/2} - 2\sqrt{u} + c = \frac{2}{3}(h+1)^{3/2} - 2\sqrt{h+1} + c$

$= \frac{2}{3}(h-2)\sqrt{h+1} + c$. Hence, $M\bar{x} = \left[\frac{20}{3}(h-2)\sqrt{1+h}\right]_0^1 = \frac{20}{3}\left(2-\sqrt{2}\right)$ and

$\bar{x} = \dfrac{\frac{20}{3}\left(2-\sqrt{2}\right)}{20\left(\sqrt{2}-1\right)} = \dfrac{\sqrt{2}}{3} \approx 0.471$ m (3 s.f.).

b Resolving vertically $T_1 + T_2 = Mg$. Taking moments about point Q, $T_1 l\cos 45^\circ = Mg\bar{x}\sin 45^\circ$ where

$l = 1$ m is the length of the rod. This gives $T_1 = Mg\dfrac{\bar{x}}{l} \approx 20\left(\sqrt{2}-1\right)\times 10 \times \dfrac{\frac{20}{3}\left(2-\sqrt{2}\right)}{20\left(\sqrt{2}-1\right)}$

$= \frac{200}{3}\left(2-\sqrt{2}\right) \approx 38.3$ N and $T_2 = Mg - T_1 \approx 42.9$ N (both 3 s.f.).

7 a The mass density of the rod is given as $m(x) = 1+3x$ kg m^{-1}, and the length $l = 10$ m. The mass of the rod is $M = \int_0^l m(x)\, dx = \int_0^{10} (1+3x)\, dx = \left[x + \frac{3}{2}x^2\right]_0^{10}$

$= 160$ kg.

The centre of mass of the rod is

$M\bar{x} = \int_0^l xm(x)\, dh = \int_0^{10} x(1+3x)\, dx$

$= \left[\frac{1}{2}x^2 + x^3\right]_0^{10} = 1050 \Rightarrow$

$\bar{x} = \frac{105}{16} \approx 6.56$ m.

Resolving vertically $N_P + N_Q = Mg$, where N_P and N_Q are the reaction forces at P and Q respectively. Taking moments about the centre of mass

$N_P(\bar{x}-1) = N_Q(l-1-\bar{x})$, which gives

$N_P = \frac{1}{8}Mg\left(9-\bar{x}\right) = \frac{1}{8}160\left(9 - \frac{105}{16}\right)g$

$= \dfrac{195}{4}g \approx 478$ N (3 s.f.), and

$N_Q = \frac{1}{8}Mg\left(\bar{x}-1\right) = \frac{1}{8}160\left(\frac{105}{16}-1\right)g$

$= \dfrac{445}{4}g \approx 1090$ N (3 s.f.)

b If the rod is on the point of turning about Q, then we take moments about point Q, noting that the distance from P to Q is eight times the distance from the mass m to Q. Hence we have

$8N_P = mg \times 1 \Rightarrow 390g = mg$

So $m = 390$ kg.

8 a If we slice the cylinder into thin horizontal slices, the mass of the cylinder is

$$M = \int_0^{30} \pi 10^2 e^{0.1h}\, dh = \pi 10^2 \left[10e^{x/10}\right]_0^{30} \text{ kg}$$
$$= \left(e^3 - 1\right)\pi$$

The centre of mass is $M\,\bar{y} = \int_0^{30} \pi 10^2 h e^{0.1h}\, dh$

$$= \pi 10^2 \left(\left[10e^{0.1h} h\right]_0^{30} - 10\int_0^{30} e^{0.1h}\, dh\right) \qquad = \pi 10^2 \left[10e^{0.1h} h - 100e^{0.1h}\right]_0^{30}$$
$$= 100\left(1 + 2e^3\right) \Rightarrow$$
$$\bar{y} = \frac{10\left(1+2e^3\right)}{e^3 - 1} \text{ cm}$$

b We find $\tan\theta = \dfrac{h - \bar{x}}{r}$, where h is the height of the cylinder and r is the radius. Hence

$$\tan\theta = \frac{30 - \bar{x}}{10} = 3 - 0.1\bar{x} = 3 - \frac{1+2e^3}{e^3 - 1}$$
$$= \frac{e^3 - 4}{e^3 - 1} \Rightarrow \theta \approx 40^\circ$$

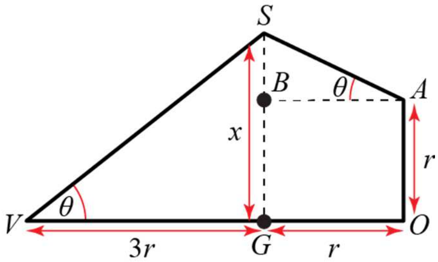
9 a The volume of the uniform solid is
$V = \pi \times 5^2 \times 10 - \frac{2}{3} \times \pi \times 3^3 = 232\pi$ cm^3. The centre of mass of the solid can be found by taking moments about point O $\pi \times 5^2 \times 10 \times 5 - \frac{2}{3} \times \pi 3^3 \times \frac{3}{8} \times 3 = 232\pi\bar{x}$
$\Rightarrow \bar{x} \approx 5.30$ cm horizontally from the point O. Note that we will want to use metric units from this point. Resolving forces vertically gives $T_1 + T_2 = Mg$. Taking the moments about the point A gives
$T_2 \times 0.1 = Mg\bar{x} \Rightarrow T_1 = Mg\left(1 - 10\bar{x}\right)$

$$= 232 \times 10^{-6} \times 10 \times \left(1 - 10 \times 0.053\right)$$
$$\approx 1.07 \times 10^{-3} \text{ N (3 s.f.) where we have taken } g = 9.8.$$

Using $T_1 + T_2 = Mg$ we then obtain that
$T_2 \approx 1.21 \times 10^{-3}$ N (3 s.f.).

b As the horizontal from the point A will be going through the centre of mass and the radius of the cylinder is 5 cm, the angle is $\tan\theta = \dfrac{\bar{x}}{5} \approx 1.06 \Rightarrow \theta \approx 46.7^\circ$.

10 a



In equilibrium the centre of mass G lies below the point of suspension S. Let distance $SG = x$.
O is the centre of the base of the cone and V is its vertex.
A and B are shown on the diagram.

$$\tan\theta = \frac{x}{3r} \left(\text{from } \Delta VSG\right)$$

$$\text{Also } \tan\theta = \frac{x-r}{r} \left(\text{from } \Delta ABS\right)$$

$$\therefore \frac{x}{3r} = \frac{x-r}{r}$$

$$\therefore x = 3x - 3r$$

$$\therefore 2x = 3r$$

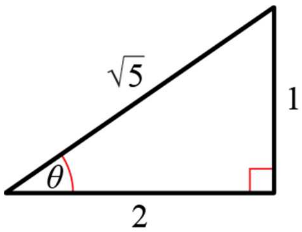
$$\therefore x = \frac{3r}{2}$$

$$\therefore \tan\theta = \frac{1}{2}$$

b Resolve vertically for the forces acting on the cone:

$$2T\sin\theta = mg$$

$$\therefore T = \frac{mg}{2\sin\theta}$$



$$\text{As } \tan\theta = \frac{1}{2}, \sin\theta = \frac{1}{\sqrt{5}} \left(\text{from Pythagoras}\right)$$

$$\therefore T = \frac{\sqrt{5}\,mg}{2} \text{ N}$$

11 First consider the metal mould. Taking moments about point O,

$$\tfrac{2}{3}\pi \times 60^3 \times \tfrac{3}{8} \times 60 - \tfrac{2}{3}\pi \times 40^3 \times \tfrac{3}{8} \times 40$$

$$= \left(\tfrac{2}{3}\pi \times 60^3 - \tfrac{2}{3}\pi \times 40^3\right)\bar{x} \Rightarrow$$

$\bar{x} = \frac{975}{38} \approx 25.7$ (3 s.f.) along the symmetry axis. Taking moments about O when the mould is filled with plastic

$$10\rho\left(\tfrac{2}{3}\pi \times 60^3 - \tfrac{2}{3}\pi \times 40^3\right)\bar{x} + \rho\left(\tfrac{2}{3}\pi \times 40^3\right) \times \tfrac{3}{8} \times 40$$

$$= \left(10\rho\left(\tfrac{2}{3}\pi \times 60^3 - \tfrac{2}{3}\pi \times 40^3\right) + \rho\left(\tfrac{2}{3}\pi \times 40^3\right)\right)\bar{X}.$$

From which we find $\bar{X} = 25.2$ cm along the symmetry axis. Now we can find the angle that the plane face makes with the vertical

$$\tan\theta = \frac{\bar{X}}{60} = 0.42 \Rightarrow \theta \approx 22.8^\circ.$$