Further centres of mass 3C

1 If the hemisphere has twice the density of the cone then the ratio of the masses becomes cone 1, hemisphere 2, composite body 3 so the moments equation becomes

$$1\times\frac{10}{4}+2\times\frac{-15}{8}=3\bar{x}$$

$$\therefore \bar{x}=\frac{-5}{12}$$

The centre of mass lies on the axis of symmetry at a point $\frac{5}{12}$ cm from O towards the rim of the hemisphere.

2 The mass of the cylinder is $M_{cyl}=\rho_{cyl}\pi 36\times 10$ and mass of the cone is $M_{cone}=\rho_{cone}\frac{1}{3}\pi 36\times 5$. We are also given that $\rho_{cyl}=3\rho_{cone}$. Suppose that the centre of mass is at a distance x above the base of the cylinder. Taking moments about the base of the cylinder gives

$$M_{cone}\left(\tfrac{1}{4}\times 5+10\right)+M_{cyl}\times 5=(M_{cone}+M_{cyl})x$$

$$x=\frac{5(9M_{cone}+4M_{cyl})}{4(M_{cone}+M_{cyl})}=\frac{5(9\times\frac{1}{3}\times 5+4\times 3\times 10)}{4(\frac{1}{3}\times 5+3\times 10)}$$

$$=\frac{405}{76}\approx 5.33 \text{ cm (3 s.f.)}.$$

3 a By symmetry, $\bar{x}=0$. If the side of the square base is a, then the area of a cross section as a function of y is $A=a^2\frac{(h-y)^2}{h^2}$ (using similar triangles), where h is the height of the pyramid. If we slice the pyramid into horizontal slices of thickness δy, mass of the pyramid is then

$$M=\rho\int_0^h A\,\mathrm{d}y=\rho\frac{a^2}{h^2}\int_0^h (h-y)^2\,\mathrm{d}y$$

$$=\rho\frac{a^2}{h^2}\left[h^2y-hy^2+\frac{y^3}{3}\right]_0^h=\frac{1}{3}\rho a^2 h, \text{ and the}$$

centre of mass

$$M\bar{y}=\rho\int_0^h yA\,\mathrm{d}y=\rho\frac{a^2}{h^2}\int_0^h y(h-y)^2\,\mathrm{d}y$$

$$=\rho\frac{a^2}{h^2}\left[\frac{h^2y^2}{2}-\frac{2hy^3}{3}+\frac{y^4}{4}\right]_0^h=\frac{1}{12}a^2h^2\rho,$$

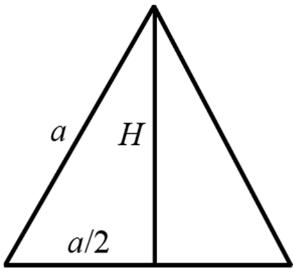
$$\Rightarrow \bar{y}=\frac{h}{4}.$$

3 b Taking the moments about the point O,

$\frac{1}{4}Mh-8^3\times 2\rho\times\frac{1}{2}\times 8=(M+8^3\times 2\rho)\bar{Y}$,

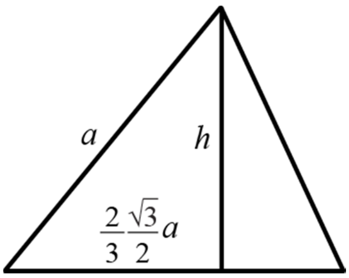
where $\bar{Y}$ is the centre of mass of the composite body. From this equation we find $\bar{Y}=\frac{61}{18}\approx 3.39\,\text{cm}$ (3 s.f.) below O.

4 a Tetrahedron is a symmetric solid, and its centre of mass will lie at the intersection of its space heights. Let the side of the tetrahedron be a. Let us first find the height of the base of the tetrahedron.



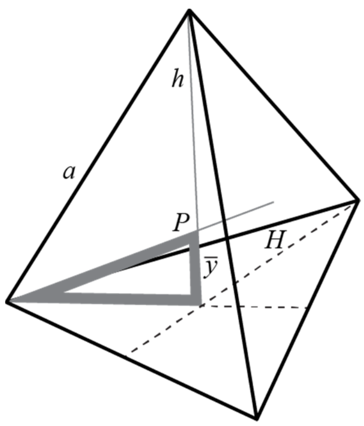
Using Pythagoras' theorem, the height of the base (or any face of the tetrahedron) is $H = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{\sqrt{3}}{2}a$.

Because the base is an equilateral triangle the heights intersect at the centre of the triangle which divide them in ratio $2:1$. The height of the tetrahedron h can be found by considering a vertical slice through the top vertex and the centre of the base, shown in a diagram below.



Using the Pythagoras theorem, $h = \sqrt{a^2 - \left(\frac{2}{3}\frac{\sqrt{3}}{2}a\right)^2} = \sqrt{\frac{2}{3}}a$.

Next consider the intersection of two such spatial heights, going from a vertex to the centre of the opposite face, at point P.



Label the shortest distance from P to a face $\bar{y}$, and this will be the height of the centre of mass above the base of the tetrahedron. Consider the bold grey triangle in the sketch above. We can find $\bar{y}$ using Pythagoras' theorem again $\bar{y}^2 + (\frac{2}{3}H)^2 = (h - \bar{y})^2$. In terms of h, $H = \frac{3}{2\sqrt{2}}h$, and solving gives us $\bar{y} = \frac{1}{4}h = \frac{a}{2\sqrt{6}}$

b Taking the moments about the point O, $M_1\bar{y} - M_2\bar{y} = (M_1 + M_2)\bar{Y}$, where M_1 and M_1 are the masses of the tetrahedrons, and $\bar{Y}$ is the centre of mass of the resulting solid. The masses are $M_1 = 3\rho\frac{a^3}{6\sqrt{2}}$ and $M_2 = \rho\frac{a^3}{6\sqrt{2}}$, thus

$$\bar{Y} = \frac{a}{2\sqrt{6}}\frac{(M_1 - M_2)}{(M_1 + M_2)} = \frac{a}{2\sqrt{6}}\frac{\left(3\rho\frac{a^3}{6\sqrt{2}} - \rho\frac{a^3}{6\sqrt{2}}\right)}{\left(3\rho\frac{a^3}{6\sqrt{2}} + \rho\frac{a^3}{6\sqrt{2}}\right)}$$

$= \frac{a}{2\sqrt{6}}\frac{(3-1)}{(3+1)} = \frac{a}{4\sqrt{6}}$, from O in the heavier tetrahedron. Given that the side length is 9 cm, $\bar{Y} = \frac{9}{4\sqrt{6}} = \frac{3\sqrt{6}}{8}$ cm below O.

5 a From question **3**, a square pyramid has its centre of mass $\frac{1}{4}h$ on the line of symmetry above its base, where h is its height. Using similar triangles we can find the height of the original pyramid $\frac{10}{h} = \frac{5}{h-5} \Rightarrow h = 10$. The volume of the pyramid is $V = \frac{1}{3}a^2h$, where a is the side of the square base. Taking the moments about the centre of the base of the pyramid, $\frac{1}{3}\times 10^3 \times \frac{10}{4} - \frac{1}{3}\times 5^3 \times \left(\frac{5}{4} + 5\right)$ $= \frac{1}{3}\times\left(10^3 - 5^3\right)\bar{y} \Rightarrow \bar{y} = \frac{55}{28} \approx 1.96$ cm.

5 b The volume of the truncated pyramid $v=\frac{1}{3}(10^3-5^3)=\frac{875}{3}$. Taking the moments about the point O,
$\rho v\left(5-\frac{55}{28}\right)-2\rho\times 5^3\times\frac{5}{2}=\left(v\rho+5^3\times 2\rho\right)\bar{Y}$
$\Rightarrow\bar{Y}=\frac{25}{52}$. Hence the centre of mass is $\bar{Y}\approx 0.481$ cm (3 s.f.) below O, towards the larger body.

6 a The mass per unit length is given as $m(x)=10\left(1-\frac{1}{12}x\right)$, and the length of the post is $l=1.2$. The total mass will be

$$M=\int_0^l m(x)\,\mathrm{d}x=\int_0^{1.2}\left(10-\frac{10}{12}x\right)\mathrm{d}x$$
$$=\left[10x-\frac{5x^2}{12}\right]_0^{1.2}=11.4\text{ kg}.$$

b The centre of mass can be found using the formula

$$M\bar{x}=\int_0^l x\,m(x)\,\mathrm{d}x=\int_0^{1.2}x\left(10-\frac{10}{12}x\right)\mathrm{d}x$$
$$=\left[5x^2-\frac{5x^3}{18}\right]_0^{1.2}=\frac{168}{25}\simeq 6.72\Rightarrow$$
$\bar{x}=\frac{56}{95}\approx 0.589$ m (3 s.f.).

c The mass would be $M=\int_0^l \pi r^2\rho(x)\,\mathrm{d}x$, where $\rho(x)=100\left(1-\frac{1}{12}x\right)$. Thus if $r=0.1$ m, the answer would change. The centre of mass would be

$$\bar{x}=\frac{\int_0^l \pi r^2 x\rho(x)\,\mathrm{d}x}{\int_0^l \pi r^2\rho(x)\,\mathrm{d}x}=\frac{\int_0^l x\left(1-\frac{1}{12}x\right)\mathrm{d}x}{\int_0^l\left(1-\frac{1}{12}x\right)\mathrm{d}x},$$
which does not depend on the extra factors, and thus would be the same as in the case **b**.

7 a The volume of the cylinder is $V=\pi(4r)^2 2r=32\pi r^3$. The volume of the hemisphere $v=\frac{2}{3}\pi r^3$. By symmetry, the centre of mass will lie in the vertical plane between O and P. Taking moments about O, $Vr-v\frac{3}{8}r=(V-v)kr\Rightarrow$

$$k=\frac{3v-8V}{8(v-V)}=\frac{381}{376}$$

For part **b** we will also need the centre of mass of the resulting solid from the axis OX. Taking the moments about OX
$V\times 0-vr=(V-v)k_H r\Rightarrow$

$k_H=\frac{v}{v-V}=-\frac{1}{47}$. The negative sign indicates that the centre of mass is away from the cavity.

b i Taking the moments about O,

$$\rho(V-v)kr+2\rho v\frac{3}{8}r$$
$$=\left(\rho V-\rho v+2\rho v\right)\bar{k}r\Rightarrow$$
$$\bar{k}=\frac{3v-4kv+4kV}{4(v+V)}=\frac{387}{392}$$

Thus the vertical distance from O will be $\frac{387}{392}r$

ii Taking the moments horizontally about the axis OX,

$$\rho(V-v)k_H r+2\rho v r$$
$$=\left(\rho V-\rho v+2\rho v\right)\bar{k}_H r\Rightarrow$$
$$\bar{k}_H=\frac{2v-k_H v+k_H V}{v+V}=\frac{1}{49}$$

Thus the horizontal distance from OX will be $\frac{1}{49}r$ in the water.

8 a Using the formula

$$\bar{x} = \frac{\int_0^l x\,m(x)\,dx}{\int_0^l m(x)\,dx} = \frac{\int_0^6 x(x+1)^2\,dx}{\int_0^6 (x+1)^2\,dx}$$

$$= \frac{\left[\frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^6}{\left[x + x^2 + \frac{x^3}{3}\right]_0^6} = \frac{486}{114} = \frac{81}{19}$$

≈ 4.26 m (3 s.f.).

b $$\bar{x} = \frac{\int_0^l x\,m(x)\,dx}{\int_0^l m(x)\,dx} = \frac{\int_0^{10} x\left(10 - \frac{1}{4}x\right)dx}{\int_0^{10} \left(10 - \frac{1}{4}x\right)dx}$$

$$= \frac{\left[5x^2 - \frac{x^3}{12}\right]_0^{10}}{\left[10x - \frac{x^2}{8}\right]_0^{10}} = \frac{\frac{1250}{3}}{\frac{175}{2}} = \frac{100}{21}$$

≈ 4.76 m (3 s.f.).

c $$\bar{x} = \frac{\int_0^l x\,m(x)\,dx}{\int_0^l m(x)\,dx} = \frac{\int_0^2 x(1+x^2)^{-1}\,dx}{\int_0^2 (1+x^2)^{-1}\,dx}$$

$$= \frac{\left[\frac{1}{2}\ln(1+x^2)\right]_0^2}{\left[\arctan x\right]_0^2} = \frac{\frac{1}{2}\ln 5}{\arctan 2}$$

≈ 0.727 m (3 s.f.).

d $$\bar{x} = \frac{\int_0^l x\,m(x)\,dx}{\int_0^l m(x)\,dx} = \frac{\int_0^5 x\,e^{0.5x}\,dx}{\int_0^5 e^{0.5x}\,dx}$$

$$= \frac{\left[2e^{x/2}x - 4e^{x/2}\right]_0^5}{\left[2e^{x/2}\right]_0^2} = \frac{4 + 6e^{5/2}}{-2 + 2e^{5/2}}$$

≈ 3.45 m (3 s.f.).

9 a The mass density of the mast is given as $m(h) = 50e^{-0.01h}$. The total mass of the mast is $M = \int_0^{18} m(h)\,dh = \int_0^{18} 50e^{-0.01h}\,dh$

$$= \left[-5000e^{-0.01h}\right]_0^{18} = 5000\left(1 - \frac{1}{e^{9/50}}\right)$$

≈ 824 kg (3 s.f.).

9 b Using the formula

$$\bar{x} = \frac{\int_0^{18} h\,m(h)\,dh}{\int_0^{18} m(h)\,dh} = \frac{\int_0^{18} h\,e^{-0.01h}\,dh}{\int_0^{18} e^{-0.01h}\,dh}$$

$$= \frac{\left[-100e^{-0.01h}h\right]_0^{18} + 100\int_0^{18} e^{-0.01h}\,dh}{\left[-100e^{-0.01h}\right]_0^{18}}$$

$$= \frac{200\left(50 - \frac{59}{e^{9/50}}\right)}{100 - \frac{100}{e^{9/50}}} \approx 8.73 \text{ m (3 s.f.).}$$

10 a The mass density is given by $m(x) = 5 - px$. The mass of the rod is

$$7 = \int_0^l m(x)\,dx = \int_0^2 (5 - px)\,dx$$

$$= \left[5x - \frac{px^2}{2}\right]_0^2 = 10 - 2p \Rightarrow p = \frac{3}{2}$$

b $$7\bar{x} = \int_0^2 x\,m(x)\,dx = \int_0^2 x(5 - px)\,dx$$

$$= \left[\frac{5x^2}{2} - \frac{px^3}{3}\right]_0^2 = 10 - \frac{8p}{3} = 6$$

$$\Rightarrow \bar{x} = \frac{6}{7} \approx 0.857 \text{ m (3 s.f.).}$$

Challenge

Given $m(x) = a(1-bx)$, the centre of mass can be found as $M\bar{x} = \int_0^l x\,m(x)\,\mathrm{d}x$, where $M = 10$ kg is the mass of the post and $l = 2$ m is the length. Integrating gives

$$10\bar{x} = \int_0^2 x\,a(1-bx)\,\mathrm{d}x = \left[\frac{ax^2}{2} - \frac{1}{3}abx^3\right]_0^2$$

$$= 2a - \frac{8ab}{3}$$

Given that $\bar{x} = 1.5$ m, $15 = 2a - \frac{8ab}{3}$

But we also have that

$$M = 10 = \int_0^l m(x)\,\mathrm{d}x = \int_0^2 a(1-bx)\,\mathrm{d}x$$

$= \left[ax - \frac{1}{2}abx^2\right]_0^2 = 2a(1-b)$. Solving this equation gives $b = 1 - \frac{5}{a}$

Substituting this into the first equation gives

$$15 = 2a - \frac{8a}{3}\left(1 - \frac{5}{a}\right) \Rightarrow 5 + 2a = 0 \Rightarrow a = -\frac{5}{2}$$

and $b = 3$