Centres of mass of plane figures Mixed exercise

1 a $\frac{\pi \times 2^2}{2} \times \left(\frac{4 \times 2}{3\pi}\right) + 2 \times \frac{3}{2} \times \left(-\frac{1}{2}\right) = \left(\frac{\pi \times 2^2}{2} + 2 \times \frac{3}{2}\right)\bar{x}$

$$\frac{16}{3} - \frac{3}{2} = (2\pi + 3)\bar{x}$$

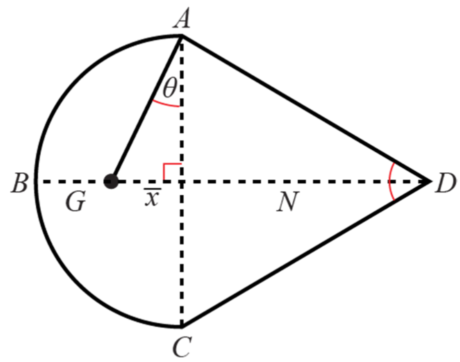
$$\frac{23}{6(2\pi + 3)} = \bar{x}$$

0.413m (3 s.f.)

Use $\sum m_i x_i = \bar{x} \sum m_i$ taking AC as the y-axis.

A decimal answer is acceptable.

b



G is the centre of mass

θ is the required angle

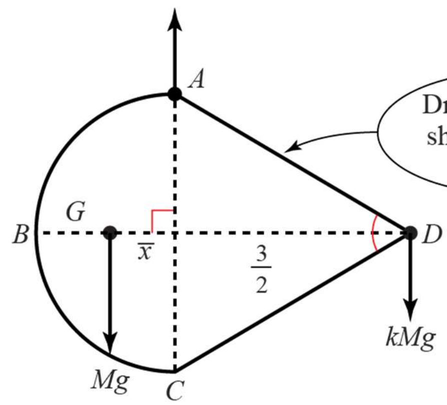
G will be on the line of symmetry.

In equilibrium, AG will be vertical.

$$\tan\theta = \frac{x}{2} = \frac{23}{12(2\pi + 3)}$$

$$\theta = 12^\circ \text{ (nearest degree)}$$

c



M(A),

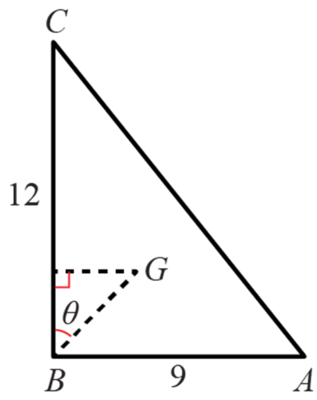
$$Mg\bar{x} = kMg \times \frac{3}{2}$$

$$\Rightarrow k = \frac{2}{3} \times \frac{23}{6(2\pi + 3)}$$

$$= \frac{23}{9(2\pi + 3)} = 0.275 \text{ (3 s.f.)}$$

Taking moments about A means we don't need to know the force A.

2



A is (9, 0)

B is (0, 0) ← Take BA and BC as axes.

C is (0, 12)

then G is (3, 4) ← Take the mean of the 3 points.

← G will be vertically below B.

In equilibrium, BG will be vertical.

Hence required angle is $G\hat{B}C = \theta$.

$\tan\theta = \frac{3}{4} \Rightarrow \theta = 36.9^\circ$.

3 $3\begin{pmatrix}1\\6\end{pmatrix} + 5\begin{pmatrix}-1\\5\end{pmatrix} + 2\begin{pmatrix}2\\-3\end{pmatrix} + 4\begin{pmatrix}-1\\-4\end{pmatrix} = (3+5+2+4)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ ← Use $\sum m_i\mathbf{r}_i = \bar{\mathbf{r}}\sum m_i$

$\begin{pmatrix}3\\18\end{pmatrix} + \begin{pmatrix}-5\\25\end{pmatrix} + \begin{pmatrix}4\\-6\end{pmatrix} + \begin{pmatrix}-4\\-16\end{pmatrix} = 14\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ ← Simplify.

$\begin{pmatrix}-2\\21\end{pmatrix} = 14\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

Hence, coordinates of the centre of mass are $\left(-\frac{1}{7}, \frac{3}{2}\right)$.

4 Taking AB and AD as axes:

$$2a^2\begin{pmatrix} a \\ \frac{1}{2}a \end{pmatrix} + 2\times\frac{1}{2}a^2\begin{pmatrix} \frac{7a}{3} \\ \frac{a}{3} \end{pmatrix} = 3a^2\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\begin{pmatrix} 2a \\ a \end{pmatrix} + \begin{pmatrix} \frac{7a}{3} \\ \frac{a}{3} \end{pmatrix} = 3\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{13a}{9} \\ \frac{4a}{9} \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

Centre of mass of the *two* triangles.

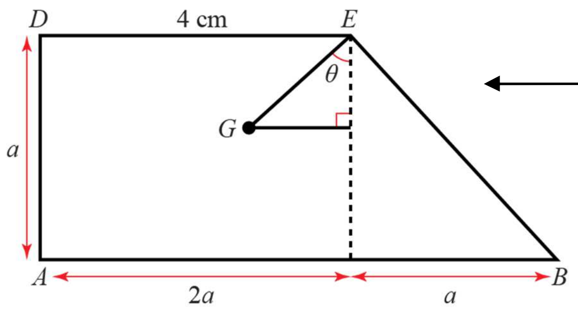
$$\frac{1}{3}\left\{\begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} 3a \\ 0 \end{pmatrix} + \begin{pmatrix} 2a \\ a \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \frac{7a}{3} \\ \frac{a}{3} \end{pmatrix}$$

a Distance from AD is $\frac{13a}{9}$

b Distance from AB is $\frac{4a}{9}$

c



EG will be vertical in equilibrium.

$D\hat{E}G$ is the angle between DE and the vertical so $(90^\circ - D\hat{E}G)$ will be the angle between DE and the horizontal.

θ is the required angle

$$\tan\theta = \frac{2a-\bar{x}}{a-\bar{y}}$$

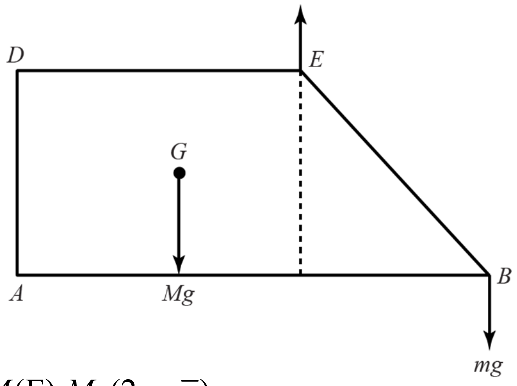
$$= \frac{2a-\frac{13a}{9}}{a-\frac{4a}{9}}$$

$$= \frac{18-13}{9-4}$$

$$= 1$$

So, θ is 45°

4 d



$$M(\mathrm{E}), Mg(2a-\bar{x}) = mga$$

$$M\frac{5a}{9} = ma$$

$$\text{i.e. } m = \frac{5M}{9}$$

Take moments about E to give an equation relating M and m.

5 a Taking axes BC and BA:

$$2a\begin{pmatrix}0\\a\end{pmatrix} + 2a\begin{pmatrix}a\\0\end{pmatrix} + a\begin{pmatrix}2a\\\frac{1}{2}a\end{pmatrix} = 5a\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}0\\2a\end{pmatrix} + \begin{pmatrix}2a\\0\end{pmatrix} + \begin{pmatrix}2a\\\frac{1}{2}a\end{pmatrix} = 5\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

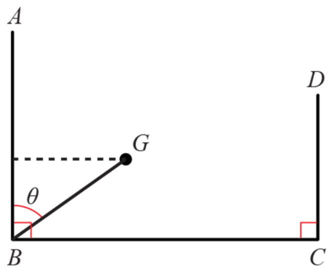
$$\begin{pmatrix}4a\\\frac{5a}{2}\end{pmatrix} = 5\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{4a}{5}\\\frac{a}{2}\end{pmatrix} = \begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Take axes through the point B, and use $\sum m_i \mathbf{r}_i = \bar{\mathbf{r}}\sum m_i$

i $\frac{4a}{5}$ **ii** $\frac{a}{2}$

b



BG will be vertical when the wire hangs in equilibrium.

θ is the required angle.

$$\tan\theta = \frac{\bar{x}}{\bar{y}}$$

$$= \frac{4a}{5} \times \frac{2}{a} = \frac{8}{3}$$

$$\Rightarrow \theta = 58^\circ \text{ (nearest degree)}$$

6 Taking AB and AD as axes:

$$6a^2\begin{pmatrix}\frac{3a}{2}\\ a\end{pmatrix} - a^2\begin{pmatrix}\frac{1}{2}a\\ \frac{1}{2}a\end{pmatrix} = 5a^2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

Treat the lamina as a rectangle with a square removed.

$$\begin{pmatrix}9a\\ 6a\end{pmatrix} - \begin{pmatrix}\frac{1}{2}a\\ \frac{1}{2}a\end{pmatrix} = 5\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}1.7a\\ 1.1a\end{pmatrix} = \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

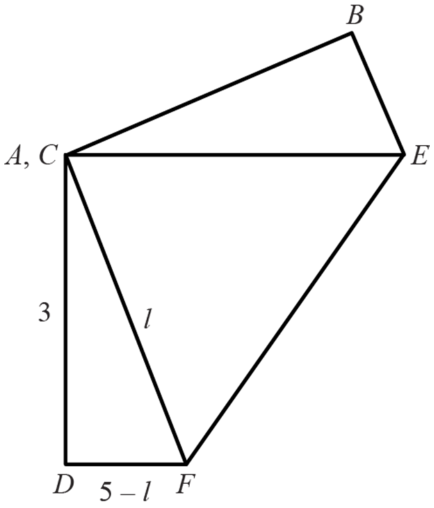
Simplify.

a $1.7a$

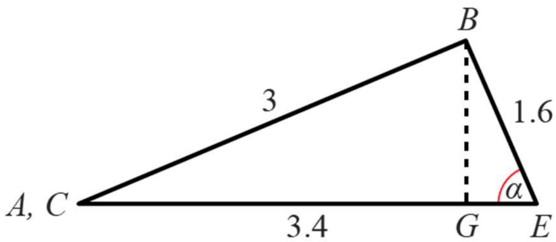
State your answers.

b $1.1a$

7



Let CF be l, therefore $DF = 5 - l$
By Pythagoras' theorem
$l^2 = 3^2 + (5-l)^2$
$l = 3.4$
Triangle ADF and ABE both have area
$\frac{1}{2} \times 3 \times 1.6 = 2.4 \text{ cm}^2$
Since total area of rectangle is 15 cm^2
$ABEFD$ has area $\frac{1}{2}(15 - 4.8) = 5.1 \text{ cm}^2$
Take D as the origin and let DF lie on the x-axis.
ADF has com at $\left(\frac{0+0+1.6}{3}, \frac{3+0+0}{3}\right) = (0.533, 1)$
CFE has com at $\left(\frac{8}{15}, 1\right)\left(\frac{0+1.6+3.4}{3}, \frac{3+0+3}{3}\right) = (1.67, 2)$
To find the coordinates of B



$\alpha = \tan^{-1}\left(\frac{3}{1.6}\right)$ $BG = 1.6\sin\alpha = 1.41$ (3 s.f.)
$AG = \sqrt{3^2 - 1.411^2} = 2.65$ (3 s.f.)
So B has coordinates (2.65, 4.41)
ABE has com at $\left(\frac{0+2.65+3.4}{3}, \frac{3+4.41+3}{3}\right) = (2.02, 3.47)$
Centre of mass of folded shape is given by

$$15\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 2.4\begin{pmatrix}0.533\\ 1\end{pmatrix} + 2.4\begin{pmatrix}1.67\\ 2\end{pmatrix} + 10.2\begin{pmatrix}2.02\\ 3.47\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1.73\\ 2.84\end{pmatrix}$$

So com lies 1.73 cm from AC and 2.84 cm from DF.

$$\tan\theta = \frac{2.84}{1.73}$$

$$\Rightarrow \theta = 59^\circ \text{ (nearest degree)}$$

8 **a** We choose coordinates so that the origin lies at A and AB lies on the x-axis then by realising the lamina as the composition of three rectangles we have that the centre of mass satisfies

$$74\begin{pmatrix}x\\y\end{pmatrix}=30\begin{pmatrix}1.5\\-5\end{pmatrix}+12\begin{pmatrix}4.5\\-8\end{pmatrix}+32\begin{pmatrix}8\\-6\end{pmatrix}$$

So

$$74\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}353\\-438\end{pmatrix}$$

Hence the angle that AH makes with the vertical satisfies

$$\tan\theta=\frac{353}{438}$$

So

$$\theta=38.9^\circ$$

b In the same coordinates as above the coordinates of the new centre of mass satisfy

$$15\begin{pmatrix}x\\y\end{pmatrix}=5\begin{pmatrix}3\\0\end{pmatrix}+\frac{10}{74}\begin{pmatrix}353\\-438\end{pmatrix}=\begin{pmatrix}62.7\\-59.2\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4.18018...\\-3.94594...\end{pmatrix}$$

Hence the angle now satisfies

$$\tan\theta=\frac{4.18018}{3.94594}$$

So

$$\theta=46.7^\circ$$

Hence the change in angle is 7.7°

9 We choose coordinates so that the origin is at B and BC lies on the x-axis then the centre of mass satisfies

$$\left(1+0.5+0.25+\tfrac{2}{3}\right)\begin{pmatrix}x\\y\end{pmatrix}=0.25\begin{pmatrix}0\\-2\end{pmatrix}$$

$$+\begin{pmatrix}3\\-4\end{pmatrix}+0.5\begin{pmatrix}5\\-2\end{pmatrix}+\frac{2}{3}\begin{pmatrix}2\\0\end{pmatrix}$$

Multiplying by 6 gives

$$14.5\begin{pmatrix}x\\y\end{pmatrix}=1.5\begin{pmatrix}0\\-2\end{pmatrix}$$

$$+6\begin{pmatrix}3\\-4\end{pmatrix}+3\begin{pmatrix}5\\-2\end{pmatrix}+4\begin{pmatrix}2\\0\end{pmatrix}$$

Which simplifies to

$$14.5\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}41\\-33\end{pmatrix}$$

Hence the angle satisfies

$$\tan\theta=\frac{41}{33}$$

$$\theta=\arctan\frac{41}{33}$$

10 a We choose coordinates so that the origin is at A and AB lies on the x-axis, then the centre of mass of the lamina satisfies

$$192\begin{pmatrix}x\\y\end{pmatrix}=92\begin{pmatrix}6\\-4\end{pmatrix}+92\begin{pmatrix}14\\-4\end{pmatrix}$$

So

$$2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}6\\-4\end{pmatrix}+\begin{pmatrix}14\\-4\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10\\-4\end{pmatrix}$$

Now taking moments about the midpoint of AB gives

$$10T_2=4\times 2Mg$$

So

$$T_2=0.8Mg$$

Taking moments about E gives

$$10T_1=6\times 2Mg$$

So

$$T_1=1.2Mg$$

b In the coordinates as above the midpoint of AB has coordinates $(6,0)$ so by considering the coordinates for the centre of mass the angle AB makes with the vertical satisfies

$$\tan\theta=\frac{4}{10-6}=1$$

So

$$\theta=45^\circ$$

11 a We choose coordinates so that the origin is at C and AB lies on the x-axis.

$$6M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=M\begin{pmatrix}-\frac{5}{2}\\ -\frac{1}{3}\times\frac{5\sqrt{3}}{2}\end{pmatrix}+2M\begin{pmatrix}0\\ -\frac{2}{3}\times\frac{5\sqrt{3}}{2}\end{pmatrix}+M\begin{pmatrix}\frac{5}{2}\\ -\frac{1}{3}\times\frac{5\sqrt{3}}{2}\end{pmatrix}+2M\begin{pmatrix}\frac{5}{2}\\ -\frac{5\sqrt{3}}{2}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{1}{6}\begin{pmatrix}5\\ -10\sqrt{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{5}{6}\\ \frac{-5\sqrt{3}}{3}\end{pmatrix}$$

Taking moments about A gives

$$10T_2=\frac{35}{6}\times 6Mg$$

$$T_2=\frac{7}{2}Mg$$

and

$$T_1=\frac{5}{2}\times Mg$$

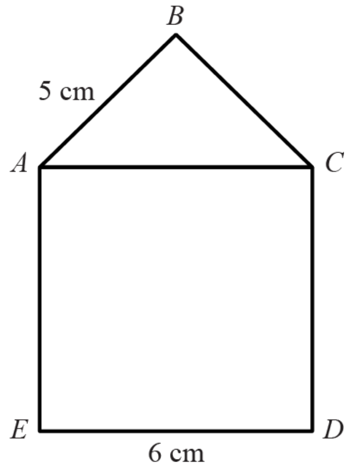
b When the string at A snaps the lamina is suspended from B and B has coordinates $(10, 0)$ so that the angle of suspension will satisfy

$$\tan\theta=\frac{\frac{25}{6}}{\frac{5\sqrt{3}}{3}}=\frac{25}{10\sqrt{3}}$$

So

$$\theta=55.3^\circ$$

Challenge



Let A be the origin and let AC lie on the x-axis. The centre of mass of the frame lies at

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\frac{5}{16}\begin{pmatrix}1.5\\ 2\end{pmatrix}+\frac{5}{16}\begin{pmatrix}4.5\\ 2\end{pmatrix}+\frac{6}{16}\begin{pmatrix}3\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}3\\ 5/4\end{pmatrix}$$

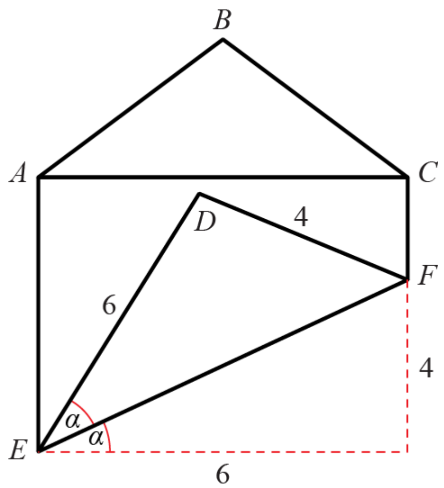
The centre of mass of the lamina is at (3, −3). So the centre of mass of the complete shape is given by

$$17M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=8M\begin{pmatrix}3\\ 5/4\end{pmatrix}+9M\begin{pmatrix}3\\ -3\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}3\\ -1\end{pmatrix}$$

Therefore $CF = 2$ cm

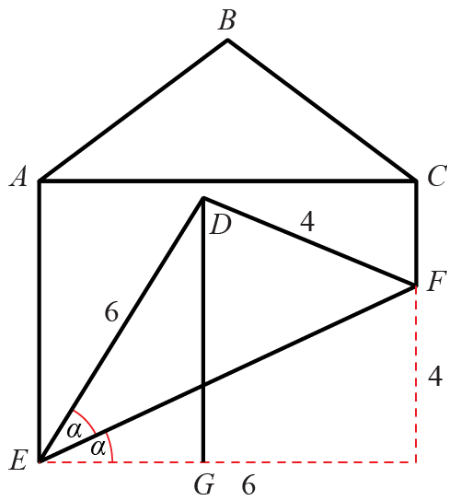
When the lamina has been folded we have



$$\tan\alpha=\frac{4}{6}\Rightarrow\alpha=33.7° \text{ (1 d.p.)}$$

Challenge (continued)

To find the coordinates of D, drop a perpendicular from D.



$\sin 2\alpha = \frac{DG}{6} \Rightarrow DG = 5.538$ cm

and

$\cos 2\alpha = \frac{EG}{6} \Rightarrow EG = 2.306$ cm

Therefore the coordinates of D are (2.306, −0.462)

Listing the coordinates of the points we have

$A(0, 0)$, $B(3, 4)$, $C(6, 0)$, $D(2.306, -0.462)$, $E(0, -6)$ and $F(6, -2)$

The com of EDF lies at $\left(\frac{0+2.306+6}{3}, \ \frac{-6+0.462-2}{3}\right) = (2.769, \ -2.821)$

EDF has area 0.5 × 6 × 4 = 12 cm^2

Therefore EDF has mass $\frac{12}{36} \times 9M = 3M$

Splitting $ACFE$ into a rectangle and a triangle, we find

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \frac{\frac{1}{2}\times 4\times 6\times \frac{1}{3}\left(\begin{pmatrix} 0 \\ -2 \end{pmatrix}+\begin{pmatrix} 0 \\ -6 \end{pmatrix}+\begin{pmatrix} 6 \\ -2 \end{pmatrix}\right)+2\times 6\times \begin{pmatrix} 3 \\ -1 \end{pmatrix}}{\frac{1}{2}\times 4\times 6+2\times 6}$$

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2.1667 \end{pmatrix}$$

$ACFE$ has mass $6M$

Challenge (continued)

Therefore the new com of the composite shape lies at

$$17M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=8M\begin{pmatrix}3\\ 1.25\end{pmatrix}+3M\begin{pmatrix}2.769\\ -2.821\end{pmatrix}+6M\begin{pmatrix}2.5\\ -2.667\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}2.783\\ -0.674\end{pmatrix}$$

So the new angle θ that AC makes with the vertical is

$$\theta=\tan^{-1}\left(\frac{0.674}{2.783}\right)$$
$$=13.6^\circ \text{ (1 d.p.)}$$