Centres of mass of plane figures 2H

1 We view the square as composed of two rectangles, with the rightmost rectangle having twice the mass of the leftmost one. Now we choose coordinates with the origin at D and DC lying on the x-axis.
Then the centre of mass of the leftmost rectangle is at $(1,2)$ and the coordinates of the rightmost is at $(3,2)$ hence the centre of mass satisfies

$$3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}+2\begin{pmatrix}3\\2\end{pmatrix}$$

Hence

$$3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}7\\6\end{pmatrix}$$

Hence

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{7}{3}\\2\end{pmatrix}$$

2 We choose coordinates with origin at D and DC lying on the x-axis, then we can view the lamina as composed of a 4 × 6 rectangle and a triangle, in these coordinates the centre of mass of the rectangle is $(8,3)$ and the centre of mass of the triangle can be obtained by averaging the coordinates of the vertices giving

$$\frac{1}{3}\begin{pmatrix}0\\6\end{pmatrix}+\frac{1}{3}\begin{pmatrix}6\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}6\\6\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}$$

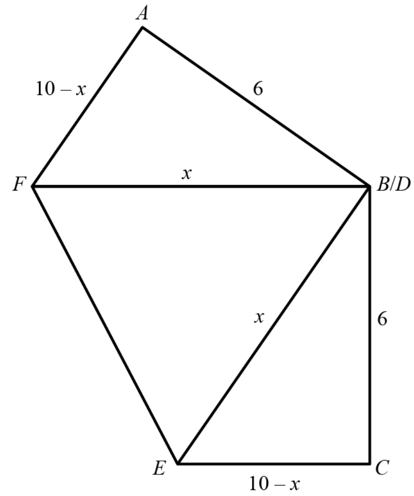
Hence the centre of mass of the lamina satisfies

$$3\begin{pmatrix}x\\y\end{pmatrix}=2\begin{pmatrix}4\\4\end{pmatrix}+\begin{pmatrix}8\\3\end{pmatrix}=\begin{pmatrix}16\\11\end{pmatrix}$$

Hence

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{16}{3}\\\frac{11}{3}\end{pmatrix}$$

3 When the lamina is folded it looks like this



Now using Pythagoras gives

$$x^2=6^2+(10-x)^2$$

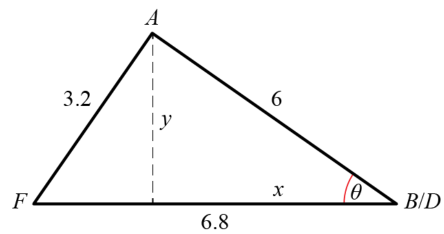
which implies that $x=6.8$. Now we choose co-ordinates with the origin at C and EC lying on the x-axis. Now the centre of mass of BCE is

$$\frac{1}{3}\begin{pmatrix}0\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-3.2\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\6\end{pmatrix}=\begin{pmatrix}-1.07\\2\end{pmatrix}$$

The centre of mass of FBE is given by

$$\frac{1}{3}\begin{pmatrix}0\\6\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-3.2\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-6.8\\6\end{pmatrix}=\begin{pmatrix}-3.33\\4\end{pmatrix}$$

Finally to find the centre of mass of the last triangle consider the following figure



We have $\sin\theta=\dfrac{3.2}{6.8}$ so that

$y=6\sin\theta=2.82$ and $x=6\cos\theta=5.29$

hence the centre of mass is given by

$$\frac{1}{3}\begin{pmatrix}0\\6\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-6.8\\6\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-5.29\\8.82\end{pmatrix}=\begin{pmatrix}-4.03\\6.94\end{pmatrix}$$

3 (continued)

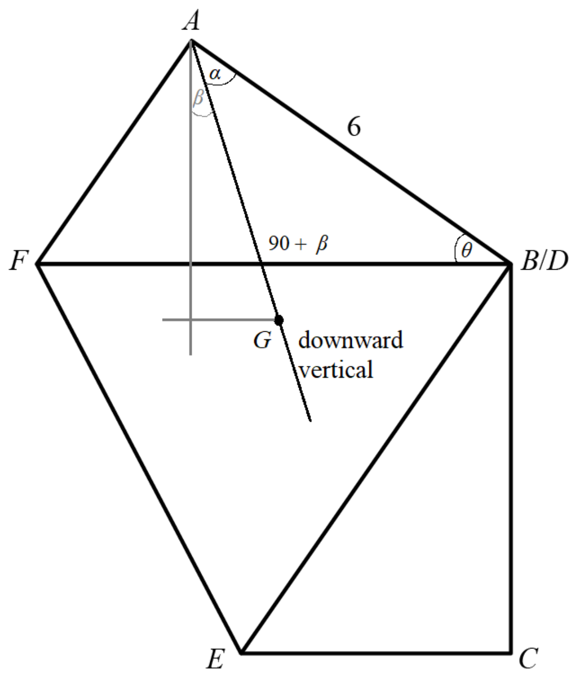
Putting this together, the centre of mass of the lamina will satisfy

$$60\begin{pmatrix}x\\y\end{pmatrix}=9.6\begin{pmatrix}-1.07\\2\end{pmatrix}+40.8\begin{pmatrix}-3.33\\4\end{pmatrix}+9.6\begin{pmatrix}-4.03\\6.94\end{pmatrix}$$

Which gives

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-3.08\\4.15\end{pmatrix}$$

The folded lamina will rest in equilibrium with its centre of mass (G) vertically below the point of suspension (A) so the required angle is α.



From earlier working,

$$\sin\theta=\frac{3.2}{6.8}\text{ so }\theta=28.1°\text{ (3 s.f.)}$$

Using trigonometry to find β:

$$\tan\beta=\frac{5.29-3.08}{8.82-4.15}=25.3°\text{ (3 s.f.)}$$

Using angles in a triangle

$$\begin{aligned}\alpha&=180°-(90°+\beta)-\theta\\&=180°-(90°+25.3°)-28.1°\\&=36.6°\text{ (3 s.f.)}\end{aligned}$$

4 From the diagram we can see that the centre of mass of AB is $(1,2)$, the centre of mass of BC is $(4,4)$, the centre of mass of CD is $(6,1)$ and the centre of mass of AD is $(3,-1)$ therefore the centre of mass of the system satisfies

$$11\begin{pmatrix}x\\y\end{pmatrix}=2\begin{pmatrix}1\\2\end{pmatrix}+3\begin{pmatrix}4\\4\end{pmatrix}+\begin{pmatrix}6\\1\end{pmatrix}+5\begin{pmatrix}3\\-1\end{pmatrix}=\begin{pmatrix}35\\12\end{pmatrix}$$

Hence

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{35}{11}\\\frac{12}{11}\end{pmatrix}$$

5 We first note that a circular wire of twice the thickness has a cross-sectional area of four times the area and so four times the mass.

From the diagram the centre of mass of AB is $(1,4)$ and its length is $\sqrt{4^2+4^2}=4\sqrt{2}$, the centre of mass of BC is $(4,2)$ and its length is $\sqrt{2^2+8^2}=2\sqrt{17}$ and the centre of mass of AC is $(2,0)$ and its length is $\sqrt{4^2+6^2}=2\sqrt{13}$ and hence the centre of mass of the system satisfies

$$\left(4\sqrt{2}+8\sqrt{13}+8\sqrt{17}\right)\begin{pmatrix}x\\y\end{pmatrix}$$
$$=4\sqrt{2}\begin{pmatrix}1\\4\end{pmatrix}+8\sqrt{17}\begin{pmatrix}4\\2\end{pmatrix}+8\sqrt{13}\begin{pmatrix}2\\0\end{pmatrix}$$

Hence

$$\left(4\sqrt{2}+8\sqrt{13}+8\sqrt{17}\right)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\sqrt{2}+16\sqrt{13}+32\sqrt{17}\\16\sqrt{2}+16\sqrt{17}\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{4\sqrt{2}+8\sqrt{13}+8\sqrt{17}}\begin{pmatrix}4\sqrt{2}+16\sqrt{13}+32\sqrt{17}\\16\sqrt{2}+16\sqrt{17}\end{pmatrix}$$
$$=(2.89,1.31)\text{ (3 s.f.)}$$

6 Firstly we can see that the mass of BC is $0.5M$ since the length of BC is half the length of AD.

Now the centre of mass of AD is $(5,1)$ the centre of mass of BC is $(5,5)$, the centre of mass of AB is $(2,3)$ and the centre of mass of CD is $(8,3)$.

Hence the centre of mass of the system satisfies

$$2.5\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\1\end{pmatrix}+0.5\begin{pmatrix}5\\5\end{pmatrix}+0.5\begin{pmatrix}2\\3\end{pmatrix}+0.5\begin{pmatrix}8\\3\end{pmatrix}$$

So

$$2.5\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}12.5\\6.5\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\2.6\end{pmatrix}$$

Now B has coordinates (3,5).

Hence the angle with the vertical satisfies

$$\tan\theta=\frac{2.4}{2}$$

Hence $\theta=50.2^\circ$

7 a We choose coordinates so that the origin is at O and AC lies on the x-axis then nothing that the length of OA is $\sqrt{1.3^2-0.5^2}=1.2$ the coordinates of the centre of mass of the triangle are given by

$$\frac{1}{3}\begin{pmatrix}0\\0.5\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\-0.5\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-1.2\\0\end{pmatrix}=\begin{pmatrix}-0.4\\0\end{pmatrix}$$

The centre of mass of the semi-circular lamina is given by

$$\begin{pmatrix}\dfrac{2\times0.5}{3}\dfrac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}\\0\end{pmatrix}=\begin{pmatrix}\dfrac{2}{3\pi}\\0\end{pmatrix}$$

Hence the centre of mass of the lamina satisfies

$$20\begin{pmatrix}x\\y\end{pmatrix}=4\begin{pmatrix}-0.4\\0\end{pmatrix}+16\begin{pmatrix}\dfrac{2}{3\pi}\\0\end{pmatrix}=\begin{pmatrix}\dfrac{32}{3\pi}-1.6\\0\end{pmatrix}$$

Hence

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\dfrac{8}{15\pi}-0.08\\0\end{pmatrix}$$

So the distance of the centre of mass from BD is simply

$$\frac{8}{15\pi}-0.08=0.09$$

b The other wire should be attached at C so that the centre of mass lies between the two wires.

c Let T_1, T_2 be the tension in the wires at B and C respectively, taking moments about B gives

$$0.5T_2=0.09\times20g$$

So

$$T_2=3.6g$$

Taking moments about C gives

$$0.5T_1=0.41\times20g$$

So

$$T_1=16.4g$$

8 a We need to find the coordinates of the centre of mass of the lamina, we choose coordinates such that the origin is the midpoint of BD and BD lies on the x-axis, by symmetry the x component of the centre of mass is zero. The centre of mass of the square is then given by $(0,-4)$.

The centre of mass of the triangle is given by

$$\frac{1}{3}\begin{pmatrix}4\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}-4\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\3\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

So the centre of mass of the lamina satisfies

$$(60\times12+20\times64)\begin{pmatrix}x\\y\end{pmatrix}=60\times12\begin{pmatrix}0\\1\end{pmatrix}$$
$$+20\times64\begin{pmatrix}0\\-4\end{pmatrix}$$

Simplifying gives

$y=-2.2$

$x=0$

So the centre of mass is $(0,-2.2)$

And coordinates of B are clearly $(-4,0)$

So the angle that AB makes with the vertical satisfies

$$\tan\theta=4\times\frac{1}{2.2}=\frac{4}{2.2}$$

So $\theta=61.2^\circ$

If we add a mass at the point A the new centre of mass will satisfy

$$(720+1280+500)\begin{pmatrix}x\\y\end{pmatrix}=720\begin{pmatrix}0\\1\end{pmatrix}$$
$$+1280\begin{pmatrix}0\\-4\end{pmatrix}+500\begin{pmatrix}-4\\-8\end{pmatrix}$$

Which simplifies to

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-0.8\\-3.36\end{pmatrix}$$

Hence the angle satisfies.

b $\tan\theta=\dfrac{3.2}{3.36}$

So

$\theta=43.6^\circ$

Challenge

a Let the width of the rectangle be x then since we fold twice, the angle of the triangle at G is 45° hence the height of the rectangle is

$$y=\frac{\frac{x}{2}}{\tan 22.5^\circ}=\frac{x}{2(\sqrt{2}-1)}=\frac{\sqrt{2}+1}{2}x$$

Which is the desired ratio.

b Split the shape into areas of ×4, ×3, ×2, ×1 layers of paper.

Using com of triangle $=\left(0,\dfrac{y_1+y_2+y_3}{3}\right)$

Find:

com of ×4 layer lies $0.667x$ below G

com of ×3 layer lies $1.138x$ below G

com of ×2 layer lies$1.61x$ below G

com of ×1 layer triangle lies $2.081x$ below G

area ×4 = $0.414x \times x = 0.414x^2$

area ×3 = $0.414x \times 0.414x = 0.171x^2$

area ×2 = = $0.585x \times 1.414x = 0.827x^2$

area ×1 triangle = $x \times x = x^2$

Let y be the position of the centre of mass below G then

$4.828x^2y = 4(0.667x \times 0.414x^2)$
$+ 3(1.138x \times 0.171x^2) + 2(1.61x \times 0.827x^2)$
$+ (2.081x \times x^2)$

$y = 1.33x$

so the centre of mass lies $1.33x$ vertically below G