Centres of mass of plane figures 2G

1 We choose coordinates so that the origin is at B and the x-axis is parallel to AC
Then the centre of mass satisfies

$$24\begin{pmatrix}x\\y\end{pmatrix}=6\begin{pmatrix}-3\\-8\end{pmatrix}+8\begin{pmatrix}0\\-4\end{pmatrix}+10\begin{pmatrix}-3\\-4\end{pmatrix}$$

So

$$24\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-48\\-120\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\-5\end{pmatrix}$$

Hence the angle BC makes with the vertical satisfies

$$\tan\theta=\frac{2}{5}$$

So $\theta=21.8^\circ$

2 We choose coordinates that are centred at D and the x-axis is parallel to BC then the centre of mass satisfies

$$50\begin{pmatrix}x\\y\end{pmatrix}=10\begin{pmatrix}-5\\5\end{pmatrix}+5\begin{pmatrix}0\\2.5\end{pmatrix}+15\begin{pmatrix}-10\\-2.5\end{pmatrix}$$
$$+5\begin{pmatrix}-7.5\\-10\end{pmatrix}+10\begin{pmatrix}-5\\-5\end{pmatrix}+5\begin{pmatrix}-2.5\\0\end{pmatrix}$$

So

$$50\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-300\\-75\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-6\\-1.5\end{pmatrix}$$

Hence the angle that CD makes with the upward vertical satisfies

$$\tan\theta=\frac{1.5}{6}=\frac{1}{4}$$

Hence the angle with the downward vertical is 104°

3 a We choose coordinates so that the origin is at A and AB lies on the x-axis, then by symmetry we have the centre of mass (a,b) satisfies

$$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}x\\-3\end{pmatrix}$$

So

$$\tan\theta=\frac{3}{x}$$

Hence $x=8\text{ cm}$

b The centre of mass of the system satisfies

$$(1+k)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\-3\end{pmatrix}+k\begin{pmatrix}0\\0\end{pmatrix}$$

Hence

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{1+k}\begin{pmatrix}8\\-3\end{pmatrix}$$

Hence the angle between the vertical and BD satisfies

$$\tan\theta=\frac{8-\frac{8}{1+k}}{\frac{3}{1+k}}=\frac{8k}{3}$$

Hence if

$$\tan\theta=\frac{8}{15}$$

We have $k=0.2$

4 We choose coordinates so that the origin is at B and the x-axis is parallel to AC
Then the centre of mass of the whole system satisfies

$$1.75M\begin{pmatrix}x\\y\end{pmatrix}=M\begin{pmatrix}-2\\-5\end{pmatrix}+0.75M\begin{pmatrix}-6\\-8\end{pmatrix}$$

So

$$1.75\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-6.5\\-11\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-\frac{26}{7}\\-\frac{44}{7}\end{pmatrix}$$

Hence the angle satisfies

$$\tan\theta=\frac{26}{44}$$

So $\theta=30.6^\circ$

5 Now we choose coordinates such that the origin is at B and BC lies on the x-axis then the centre of mass of the original lamina using Question **2** is

$$\begin{pmatrix} -6 \\ -1.5 \end{pmatrix} + \begin{pmatrix} 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 \\ -6.5 \end{pmatrix}$$

Now when the particle is attached to F the new centre of mass satisfies

$$1.15M\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} 4 \\ -6.5 \end{pmatrix} + 0.15M\begin{pmatrix} 5 \\ -15 \end{pmatrix}$$

So

$$1.15\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4.75 \\ -8.75 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{95}{23} \\ -\frac{175}{23} \end{pmatrix}$$

Hence the angle satisfies

$$\tan\theta = \frac{175}{95}$$

So $\theta = 61.5^\circ$

6 Choose coordinates such that the origin is at A and FB is parallel to the x-axis

Then the centre of mass of the top circular segment is

$$\begin{pmatrix} \frac{16}{3\pi} \\ -4+\frac{16}{3\pi} \end{pmatrix}$$

The centre of mass of the square is

$$\begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

And the centre of mass of the bottom circular segment is

$$\begin{pmatrix} 4-\frac{16}{3\pi} \\ -8-\frac{16}{3\pi} \end{pmatrix}$$

Hence the centre of mass of the whole lamina satisfies

$$(16+8\pi)\begin{pmatrix} x \\ y \end{pmatrix} = 4\pi\begin{pmatrix} \frac{16}{3\pi} \\ -4+\frac{16}{3\pi} \end{pmatrix}$$

$$+4\pi\begin{pmatrix} 4-\frac{16}{3\pi} \\ -8-\frac{16}{3\pi} \end{pmatrix} + 16\begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

Hence

$$(16+8\pi)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16\pi+32 \\ -96-48\pi \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{16+8\pi}\begin{pmatrix} 16\pi+32 \\ -96-48\pi \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

Hence the angle between the vertical and FE satisfies

$$\tan\theta = \frac{2}{6}$$

So $\theta = 18.4^\circ$

7 a Choose coordinates with the origin at A and x-axis parallel to FE then the centre of mass of the framework satisfies

$$40\begin{pmatrix}x\\y\end{pmatrix}=12\begin{pmatrix}0\\-6\end{pmatrix}+8\begin{pmatrix}4\\-12\end{pmatrix}+4\begin{pmatrix}8\\-10\end{pmatrix}$$
$$+5\begin{pmatrix}5.5\\-8\end{pmatrix}+8\begin{pmatrix}3\\-4\end{pmatrix}+3\begin{pmatrix}1.5\\0\end{pmatrix}$$

Which simplifies to

$$40\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\begin{pmatrix}120\\-280\end{pmatrix}$$

so

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\begin{pmatrix}3\\-7\end{pmatrix}$$

Taking moments about A then gives

$$8T_2=3W$$

So

$$T_2=\frac{3W}{8}$$

Taking moments about D then gives

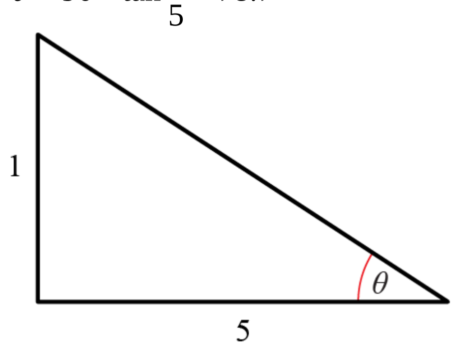
$$8T_1=(8-3)W=5W$$

So

$$T_1=\frac{5W}{8}$$

b Considering the coordinates for the centre of mass and the diagram the angle will satisfy

$$\theta=90-\tan\frac{1}{5}=78.7°$$



8 a We choose coordinates so that the origin is at A and AB lies on the x-axis then the centre of mass of the framework satisfies

$$42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=6\begin{pmatrix}3\\0\end{pmatrix}+3\begin{pmatrix}7.5\\2\end{pmatrix}+10\begin{pmatrix}9\\-3\end{pmatrix}$$
$$+3\begin{pmatrix}7.5\\-8\end{pmatrix}+10\begin{pmatrix}6\\-3\end{pmatrix}+6\begin{pmatrix}3\\-4\end{pmatrix}+4\begin{pmatrix}0\\-2\end{pmatrix}$$

So

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\frac{1}{42}\begin{pmatrix}231\\-110\end{pmatrix}=\begin{pmatrix}231/42\\-110/42\end{pmatrix}$$

Now taking moments about A gives

$$9T_2\sin 30^\circ=\frac{231}{42}W$$

Hence

$$T_2=\frac{11W}{9}$$

And taking moments about D gives

$$9T_1=\frac{147}{42}W$$

So

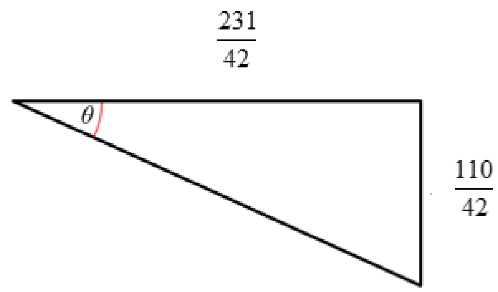
$$T_1=\frac{7W}{18}$$

b By considering the coordinates for the centre of mass, the angle satisfies

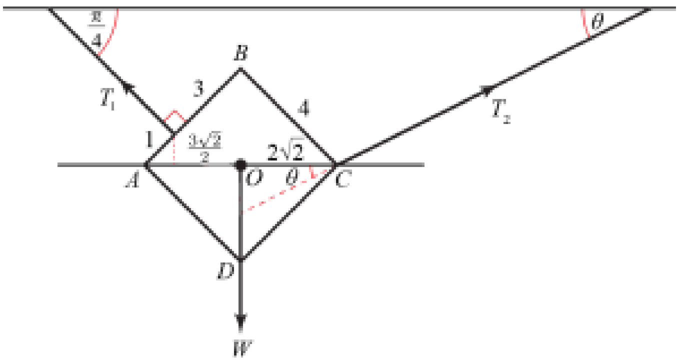
$$\tan\theta=\frac{110}{231}$$

So

$$\theta=25.5°$$



Challenge



Origin O at centre of mass of framework.
AC lies on x-axis.
Resolve forces in x direction.

$$T_1 \cos\left(\frac{\pi}{4}\right) = T_2 \cos\theta$$

$$\cos\theta = \frac{T_1}{\sqrt{2}T_2}$$

The angle will be the same for any size of square, so we can let the side of the square be 4 to make calculations straightforward.
Taking moments about O:

$$\frac{3\sqrt{2}}{2} T_1 \sin(45°) = 2\sqrt{2}T_2 \sin\theta$$

$$\frac{3}{2}T_1 = 2\sqrt{2}T_2 \sin\theta$$

$$\sin\theta = \frac{3T_1}{4\sqrt{2}T_2}$$

$$\tan\theta = \frac{3T_1}{4\sqrt{2}T_2} \times \frac{\sqrt{2}T_2}{T_1} = \frac{3}{4}$$

$$\theta = 36.9°$$