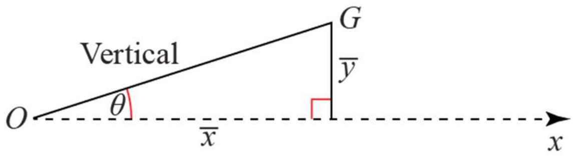
Centres of mass of plane figures 2F

1 a From question **1a** in Exercise 2D,

$\bar{x} = 2\frac{1}{2}; \bar{y} = \frac{13}{14}$

Vertical



In equilibrium, G will be vertically below O i.e. OG is the vertical.

$\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{\frac{13}{14}}{\frac{5}{2}}$

$= \frac{13}{14} \times \frac{2}{5} = \frac{13}{35}$

$\theta = \tan^{-1}\left(\frac{13}{35}\right) = 20.4° \text{ (3 s.f.)}$

b From question **1a** in Exercise 2E,

$\bar{x} = \frac{11}{4}; \bar{y} = \frac{5}{4}$

As above, $\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{\frac{5}{4}}{\frac{11}{4}}$

i.e. $\tan\theta = \frac{5}{11}$

$\theta = \tan^{-1}\left(\frac{5}{11}\right) = 24.4° \text{ (3s.f.)}$

c From question **1b** in Exercise 2D.

$\bar{x} = 1.7; \bar{y} = 2.6$

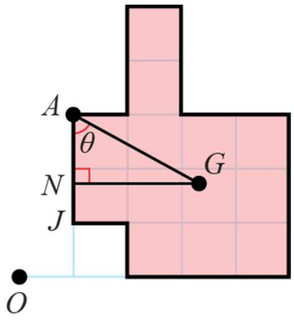
$\tan\theta = \frac{2.6}{1.7} = \frac{26}{17}$

$\theta = \tan^{-1}\left(\frac{26}{17}\right) = 56.8° \text{ (3 s.f.)}$

2 From question **4** in Exercise 2D,

$\bar{x} = \frac{79}{26}$; $\bar{y} = \frac{51}{26}$

These are the coordinates of the centre of mass, G, referred to O as origin.



A is the point of suspension.

G is $\left(\frac{79}{26}, \frac{51}{26}\right)$.

When the lamina hangs in equilibrium from A, AG will be the downward vertical.

Let N be the point on AJ such that GN is perpendicular to AJ.

See diagram.

Then $N\hat{A}G = \theta$ is the required angle.

$$\tan\theta = \frac{GN}{AN} = \frac{\bar{x} - 1}{3 - \bar{y}}$$

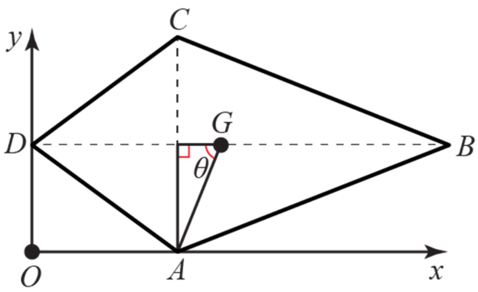
Since A is the point (1, 3).

$$= \frac{\frac{79}{26} - 1}{3 - \frac{51}{26}} = \frac{79 - 26}{78 - 51}$$

Multiply top and bottom by 26.

$$= \frac{53}{27} \Rightarrow \theta = 63.0^\circ \text{ (3 s.f.)}$$

3



G, the centre of mass has coordinates $\left(\frac{7}{3}, 2\right)$ taking O as origin.

Since AG will be *vertical* in equilibrium, the angle between AC and the *horizontal* will be θ.

θ is the required angle

$$\tan\theta = \frac{2}{\frac{7}{3} - 2}$$

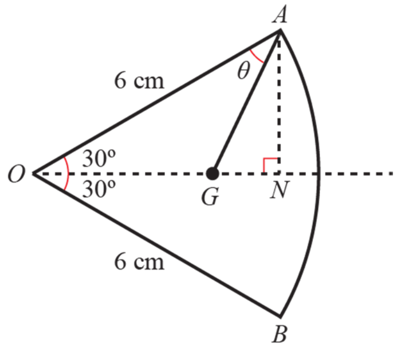
Multiply top and bottom by 3 to clear fractions.

$$= \frac{6}{7 - 6}$$

$$= 6$$

$$\theta = 80.5^\circ \text{ (3 s.f.)}$$

4



G is the centre of mass if the framework

$$OG = \bar{x} = \frac{9(\sqrt{3}+2)}{(6+\pi)}$$

G is on the line of symmetry.

(see question **2** in Exercise 2E)

θ (see diagram) is the required angle.

AG will be vertical, when the framework hangs in equilibrium.

$\theta = 60^\circ - G\hat{A}N$

$$\tan G\hat{A}N = \frac{GN}{AN} = \frac{6\cos 30^\circ - \bar{x}}{6\sin 30^\circ}$$

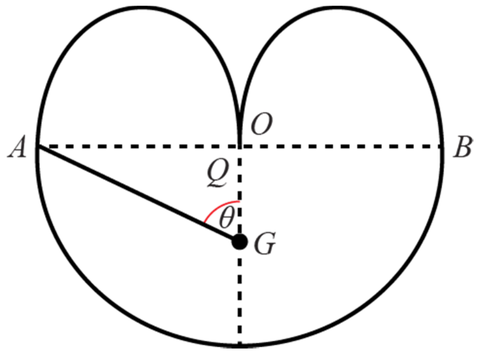
$$= \frac{3\sqrt{3} - \bar{x}}{3}$$

$$= \sqrt{3} - \frac{3(\sqrt{3}+2)}{6+\pi}$$

So, $G\hat{A}N = 26.898°\ldots$

So, $\theta = 33.1°$ (3 s.f.)

5



$OG = \dfrac{3}{2\pi}$, where *G* is the centre of mass so, *AG* will be vertical in equilibrium.

See question 4 in Exercise 2E.

Since the angle with the horizontal will be 90° – angle with the vertical.

Required angle is $A\hat{G}O = \theta$

$$\tan\theta = \frac{AO}{OG}$$

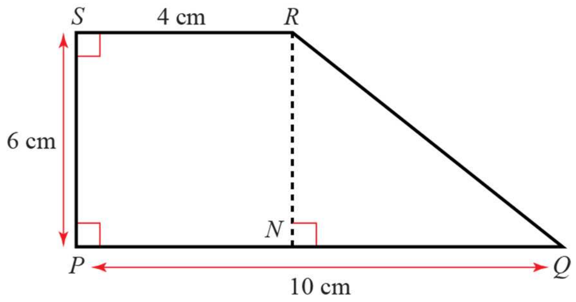
$$= \frac{3}{\frac{3}{2\pi}}$$

$$= 2\pi$$

$$\theta = \tan^{-1}(2\pi)$$

$$= 81.0° \text{ (3 s.f.)}$$

6



Centre of mass of ΔRNQ has position vector

Taking PQ and PS as axes.

$$\frac{1}{3}\left\{\begin{pmatrix}4\\0\end{pmatrix}+\begin{pmatrix}10\\0\end{pmatrix}+\begin{pmatrix}4\\6\end{pmatrix}\right\}=\begin{pmatrix}6\\2\end{pmatrix}$$

$$24\begin{pmatrix}2\\3\end{pmatrix}+18\begin{pmatrix}6\\2\end{pmatrix}=42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$$\begin{pmatrix}48\\72\end{pmatrix}+\begin{pmatrix}108\\36\end{pmatrix}=42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}156\\108\end{pmatrix}=42\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}\frac{26}{7}\\\frac{18}{7}\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

So the centre of mass of the lamina is

a $\frac{26}{7}$ cm from PS and

b $\frac{18}{7}$ cm from PQ.

c We have

$\tan\theta=\left(\frac{18}{7}\right)/\left(10-\frac{26}{7}\right)=\frac{9}{22}$

So $\theta=22.2°$

7 We choose coordinates so that the origin is at C and that BC lies on the x-axis, by considering the lamina as the union of a 2 × 6 rectangle and a 6 × 2 rectangle we see the centre of mass will satisfy

$$24\begin{pmatrix} x \\ y \end{pmatrix} = 12\begin{pmatrix} -1 \\ -3 \end{pmatrix} + 12\begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Now we attach a mass of 0.2M kg to F so the centre of mass of the whole system will satisfy

$$1.2M\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} 1 \\ -4 \end{pmatrix} + 0.2M\begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

So

$$1.2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2.2 \\ -5.2 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 11 \\ -26 \end{pmatrix}$$

Hence the angle θ satisfies

$$\tan\theta = \frac{26}{11}$$

So

$$\theta = 67.1°$$

8 We choose coordinates so that the origin is at B and the x-axis is parallel to AC then the centre of mass of the lamina is

$$\begin{pmatrix} 0 \\ -\frac{32}{3\pi}\sin\frac{\pi}{4} \end{pmatrix}$$

And the coordinates of C are

$$\begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

Hence the centre of mass of the system
Satisfies

$$1.5M\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} 0 \\ -\frac{32}{3\pi}\sin\frac{\pi}{4} \end{pmatrix} + 0.5M\begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2}{3}\begin{pmatrix} \sqrt{2} \\ -\frac{16\sqrt{2}}{3\pi} - \sqrt{2} \end{pmatrix} = \frac{2\sqrt{2}}{3}\begin{pmatrix} 1 \\ -\frac{(16+3\pi)}{3\pi} \end{pmatrix}$$

$$\simeq \begin{pmatrix} 0.9428... \\ -2.5434... \end{pmatrix}$$

Hence

$$\theta = 45° + \tan^{-1}\left(\frac{0.9428}{2.5434}\right) = 45° + 20.3° = 65.3°$$

9 a By considering the lamina as a union of two rectangles of size 2 × 4 and 4 × 2 we have the distance of the centre of mass to AC satisfies

$16x = 8 \times 2 + 8 \times 5$

So

$16x = 56$

$x = \frac{7}{2}$

b Let T be the tension in the string at B then taking moments about A gives

$6T = \frac{7}{2} \times 12g$

So

$T = 7g$

Now let T be the tension in the string at A then taking moments about B gives

$6T = \frac{5}{2} \times 12g$

So

$T = 5g$

c We will need to find the distance of the centre of mass from AB, by considering the rectangles that make up the lamina we have

$16y = 8 \times 1 + 8 \times 2$

So

$y = 1.5$

Hence the angle with the vertical is

$\tan \theta = \frac{1.5}{2.5}$

So $\theta = 31.0$

Hence the angle with the horizontal is

$\varphi = 180° - \theta = 149°$

10 We choose coordinates so that the origin is at the midpoint of PQ and the y-axis is parallel to PQ, then the coordinates of the centre of mass is

$$\left(\frac{4r}{3\pi}, 0\right)$$

So after attaching the mass at Q the centre of mass of the whole system satisfies

$$(1+k)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4r}{3\pi} \\ 0 \end{pmatrix} + k\begin{pmatrix} 0 \\ -2r \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+k}\begin{pmatrix} \frac{4r}{3\pi} \\ -2rk \end{pmatrix}$$

Now taking moments about P and the point where the right string meets the semicircle gives

$$rT_2 = \frac{4r}{3\pi(1+k)} \times (1+k)m$$

$$rT_1 = \left(r - \frac{4r}{3\pi(1+k)}\right) \times (1+k)m$$

Now using $T_1 = 5T_2$ gives

$$T_2 = \frac{4m}{3\pi}$$

$$5T_2 = \left(1 - \frac{4}{3\pi(1+k)}\right) \times (1+k)m$$

So

$$\frac{20}{3\pi} = \left(1 - \frac{4}{3\pi(1+k)}\right) \times (1+k)$$

So

$$k = \frac{8}{\pi} - 1$$

11 a We take A to be the origin, then the lamina can be seen as a rectangle with a circular section removed

we have the centre of mass satisfies $\left(6a^2-\frac{\pi}{4}a^2\right)\begin{pmatrix}x\\y\end{pmatrix}=6a^2\begin{pmatrix}a\\\frac{3a}{2}\end{pmatrix}-\frac{\pi}{4}a^2\begin{pmatrix}2a-\frac{2a\sin\frac{\pi}{4}}{3\left(\frac{\pi}{4}\right)}\cos\frac{\pi}{4}\\\frac{2a\sin\frac{\pi}{4}}{3\left(\frac{\pi}{4}\right)}\sin\frac{\pi}{4}\end{pmatrix}$

So

$$\left(6a^2-\frac{\pi}{4}a^2\right)\begin{pmatrix}x\\y\end{pmatrix}=6a^2\begin{pmatrix}a\\\frac{3a}{2}\end{pmatrix}-\frac{\pi}{4}a^2\begin{pmatrix}2a-\frac{4a}{3\pi}\\\frac{16a}{3\pi}\end{pmatrix}$$

So we have

$$\left(6-\frac{\pi}{4}\right)\begin{pmatrix}x\\y\end{pmatrix}=6\begin{pmatrix}a\\\frac{3a}{2}\end{pmatrix}-\frac{\pi}{4}\begin{pmatrix}2a-\frac{4a}{3\pi}\\\frac{4a}{3\pi}\end{pmatrix}$$

So

$$\frac{24-\pi}{4}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}6a\\9a\end{pmatrix}-\begin{pmatrix}\frac{a\pi}{2}-\frac{a}{3}\\\frac{a}{3}\end{pmatrix}$$

So

$$(24-\pi)\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{3}\begin{pmatrix}76a-6a\pi\\104a\end{pmatrix}$$

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{3(24-\pi)}\begin{pmatrix}76a-6a\pi\\104a\end{pmatrix}$$

As required.

b Let T_1, T_2 be the tensions in left and right strings respectively, taking moments about B gives

$$2aT_2=\frac{2a(38-3\pi)}{3(24-\pi)}\times W$$

Hence

$$T_2=\frac{(38-3\pi)W}{3(24-\pi)}$$

Now taking moments about C gives

$$2aT_1=\left(2a-\frac{2a(38-3\pi)}{3(24-\pi)}\right)W$$

Hence

$$T_1=\left(1-\frac{(38-3\pi)}{3(24-\pi)}\right)W=\frac{34W}{3(24-\pi)}$$

c We have the angle θ satisfies

$$\tan\theta=\left(2a-\frac{2a(38-3\pi)}{3(24-\pi)}\right)\left(3a-\frac{104a}{3(24-\pi)}\right)^{-1}$$

Hence

$$\tan\theta=\left(2-\frac{2(38-3\pi)}{3(24-\pi)}\right)\left(3-\frac{104}{3(24-\pi)}\right)^{-1}$$

Which gives

$\theta=39.1^\circ$

Challenge

We choose coordinates such that the origin is where the string meets the mobile and AH is parallel to the x-axis then by symmetry the centre of mass of the mobile lies on the line from O to the midpoint of AH and the y component satisfies

$(50+50+300)y = 50\times 5+50\times 5$

$+300\times -2.5$

So $400y = -250$

So

$y = -\frac{5}{8}$

Now the coordinates of A are $(-30,-5)$

So the centre of mass of the system satisfies

$$(M+m)\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} 0 \\ -\frac{5}{8} \end{pmatrix} + m\begin{pmatrix} -30 \\ -5 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{8(M+m)}\begin{pmatrix} -240m \\ -5M-40m \end{pmatrix}$$

Now we consider the angle that the line DE makes with the vertical this satisfies

$$\tan\theta = \frac{5M+40m}{240m}$$

On the other hand, if G touches the ceiling then the triangle formed by O, G and the point where the cable meets the ceiling gives

$$\tan\theta = \frac{\sqrt{375}}{25}$$

So

$$\frac{5M+40m}{240m} = \frac{\sqrt{375}}{25} = \frac{\sqrt{15}}{5}$$

Which gives

$m = 0.0343M$