Centres of mass of plane figures 2E

1 a

$$5\begin{pmatrix}2.5\\0\end{pmatrix}+3\begin{pmatrix}5\\1.5\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+\begin{pmatrix}3\\2.5\end{pmatrix}+3\begin{pmatrix}1.5\\2\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

There are other ways of splitting the lamina up (*see below*).

$$\begin{pmatrix}43\\21\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\frac{43}{16}=\bar{x}$$

$$\frac{21}{16}=\bar{y}$$

Equate **i** and **j** components.

Centre of mass is $\left(\frac{43}{16},\frac{21}{16}\right)$

b

$$3\begin{pmatrix}3.5\\0\end{pmatrix}+3\begin{pmatrix}5\\1.5\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+2\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}2.5\\5\end{pmatrix}+2\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}1.5\\3\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1.5\\1\end{pmatrix}+\begin{pmatrix}2\\0.5\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$$\begin{pmatrix}53\\40\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Collect terms.

$$\frac{53}{18}=\bar{x};\ \frac{40}{18}=\bar{y}$$

Centre of mass is $\left(\frac{53}{18},\frac{20}{9}\right)$

You could use decimals.

c

$$\begin{pmatrix}0.5\\2\end{pmatrix}+2\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}2\\0\end{pmatrix}+2\begin{pmatrix}3\\1\end{pmatrix}+3\begin{pmatrix}4.5\\2\end{pmatrix}+5\begin{pmatrix}4\\3.5\end{pmatrix}+\sqrt{13}\begin{pmatrix}1\\3.5\end{pmatrix}=\left(15+\sqrt{13}\right)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$$\begin{pmatrix}49.61\\42.12\end{pmatrix}=18.61\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $(2.67,\ 2.26)$

d By symmetry, $\bar{y}=3$

x-coordinate of the centre of mass satisfies

$$\sqrt{29}\times3.5+\sqrt{20}\times4+\sqrt{29}\times3.5+\sqrt{20}\times4=2\left(\sqrt{20}+\sqrt{29}\right)\times\bar{x}$$

Find the mean of the x-coordinates of the vertices.

$$7\sqrt{29}+8\sqrt{20}=2\left(\sqrt{20}+\sqrt{29}\right)\bar{x}$$

$$3.73=\bar{x}$$

Centre of mass is (3.27, 3)

2 $AB = (12+2\pi) - 12 = 2\pi$

Let $A\hat{O}B = \theta\,(\text{radians})(=2\alpha)$

Then

$$\theta = \frac{2\pi}{6} = \frac{\pi}{3} = 60^\circ$$

Distance of G from O

$= \bar{x}$ say.

Then,

$$(6\times 3\cos 30^\circ)\times 2 + 2\pi \times \frac{6\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \bar{x}(12+2\pi)$$

$$18\sqrt{3} + 36 = \bar{x}(12+2\pi)$$

$$\therefore \quad \bar{x} = \frac{18\sqrt{3}+36}{12+2\pi}$$

$$= \frac{9(\sqrt{3}+2)}{6+\pi}$$

Use $S = r\theta$.

$\frac{r\sin\alpha}{\alpha}$; α in

from the formula booklet.

Use $\sum m_i x_i = \bar{x}\sum m_i$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{6} = \frac{1}{2}$

Simplify by dividing top and bottom by 2.

Centre of mass is on line of symmetry through O,

and a distance of $\frac{9(\sqrt{3}+2)}{6+\pi}$ from O.

State your answer.

3 We choose coordinates with the midpoint of AB as the origin and AB lying on the x-axis, then the distance from AB to the centre of mass is just the modulus of the y-coordinate of the centre of mass since by symmetry the centre of mass lies on the y-axis, let the centre of mass have coordinates $(0, y)$ then since the total mass of the system is $3a+a+a+a+a+3a+3a+a=14a$ we have

$$14ay = 3a\times 0 + a\times\left(-\tfrac{1}{2}a\right) + a\times\left(-\tfrac{1}{2}a\right)$$
$$+a\times(-a) + a\times(-a)$$
$$+3a\times\left(-\tfrac{5}{2}a\right) + 3a\times\left(-\tfrac{5}{2}a\right)$$
$$+a\times(-4a)$$

Simplifying gives

$$14ay = -a^2 - 2a^2 - 15a^2 - 4a^2 = -22a^2$$

Hence

$$y = -\frac{11a}{7}$$

So the distance to AB is $\frac{11a}{7}$

4 a We choose coordinates so that the origin is at O and AB is parallel to the x-axis, then the centre of mass of the straight piece of wire is O and its mass is 30.
For the circular piece, the mass is 15π and its centre of mass is at

$$\left(0, \frac{15\sin\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)}\right) = \left(0, \frac{30}{\pi}\right)$$

Now by symmetry the centre of mass of the system lies on the y-axis and the y-coordinate will satisfy

$$y = \frac{450}{30+15\pi} = 5.83 \text{ to 3 s.f.}$$

And by our choice of coordinates this is precisely the distance to AB.

b i The total mass is $M = 100 + 100 + 8(30+15\pi) = 440 + 120\pi = 817$ g to 3 s. f.

ii We keep the same coordinate system as before, so that the centre of mass still lies on the y-axis by symmetry hence the new y-coordinate satisfies

$$(440+120\pi)\,y = 120\pi \times \frac{30}{\pi} = 3600$$

$$y = \frac{3600}{440+120\pi} = 4.41 \text{ to 3 s.f.}$$

and as before this is precisely the distance to AB.

5 We choose coordinates such that the origin lies at A and AB lies on the x-axis, we start by finding the coordinates of the centre of mass of the unloaded framework. Its coordinates will satisfy

$$15\begin{pmatrix} x \\ y \end{pmatrix} = \frac{3\times15}{12}\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix} + \frac{4\times15}{12}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$+\frac{5\times15}{12}\begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix}$$

That is $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{3}{12}\begin{pmatrix} \frac{3}{2} \\ 0 \end{pmatrix} + \frac{4}{12}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \frac{5}{12}\begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix}$

So $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{12}\begin{pmatrix} \frac{9}{2}+12+\frac{15}{2} \\ -8-10 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 24 \\ -18 \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix}$

Now we consider the whole system, the total mass is $M = 15 + 10 + 20 + 30 = 75$
And so the coordinates of the centre of mass satisfy

$$75\begin{pmatrix} x \\ y \end{pmatrix} = 10\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 20\begin{pmatrix} 3 \\ 0 \end{pmatrix} + 30\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$+15\begin{pmatrix} 2 \\ -\frac{3}{2} \end{pmatrix}$$

Which simplifies to $75\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 180 \\ \frac{-285}{2} \end{pmatrix}$

So $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ -\frac{19}{10} \end{pmatrix}$

6 We choose coordinates such that the is at O and the x-axis is parallel to AB then the centre of mass of the top left semicircle is $\left(-1.5, \frac{3}{\pi}\right)$ and by symmetry the centre of mass of the top right semicircle is $\left(1.5, \frac{3}{\pi}\right)$

Finally, the centre of mass of the larger semicircle is $\left(0, -\frac{6}{\pi}\right)$ hence the centre of mass of the framework is given by

$$6\pi\begin{pmatrix} x \\ y \end{pmatrix} = 3\pi\begin{pmatrix} 0 \\ -\frac{6}{\pi} \end{pmatrix} + 1.5\pi\begin{pmatrix} -1.5 \\ \frac{3}{\pi} \end{pmatrix} + 1.5\pi\begin{pmatrix} 1.5 \\ \frac{3}{\pi} \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -\frac{6}{\pi} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -1.5 \\ \frac{3}{\pi} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1.5 \\ \frac{3}{\pi} \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4\pi}\begin{pmatrix} 0 \\ -6 \end{pmatrix} = \frac{1}{2\pi}\begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

7 **a** The ladder has a line of symmetry that is parallel to the rungs and between the 3rd and 4th rungs hence the height of the centre of mass is $3 \times 50 + 25 = 175\text{ cm} = 1.75\text{ m}$.

b Choose coordinates so that the origin is at the centre of mass and the rungs are parallel to the x-axis, the total mass of the whole ladder including the bottom rung is then proportional to 20 the number of lengths of 50cm it contains, and the mass of the bottom run is proportional to 1 hence in this coordinate system the new centre of mass satisfies

$$19\begin{pmatrix} x \\ y \end{pmatrix} = 20\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -125 \end{pmatrix}$$

Hence

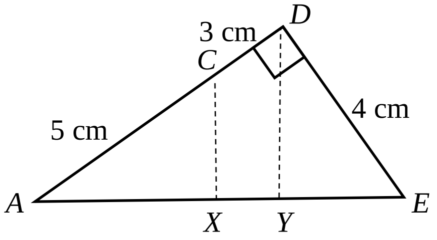
$$y = \frac{125}{19}\text{cm} = \frac{5}{76}\text{m}$$

Challenge

AB = 4 cm and CD = 3 cm, so AC = 5 cm

By symmetry the centre of mass lies on the straight line that goes between C and the midpoint of AE so also by symmetry the y-coordinate of the centre of mass is the same as the y-coordinate of the centre of mass of one of the right-angled triangles.

Consider triangle AED



By Pythagoras, $AE = \sqrt{(5+3)^2 + 4^2} = \sqrt{80} = 4\sqrt{5}$ cm

$$AX = \frac{AE}{2} = 2\sqrt{5}$$

Using Pythagoras in triangle ACX

Height of C = $CX = \sqrt{5^2 - \left(2\sqrt{5}\right)^2} = \sqrt{5}$

Let θ be the angle AED, then we have

$$\tan\theta = \frac{8}{4} = 2$$

$$\sin\theta = \frac{8}{4\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\theta = \frac{4}{4\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Consider triangle YDE

Height of D = height of B = $DY = 4\sin AED = \dfrac{8\sqrt{5}}{5}$

Find y-coordinate of centre of mass of ABC:

Taking A as origin

y-coordinate for centre of mass of AB = $\dfrac{1}{2}DY = \dfrac{4\sqrt{5}}{5}$

y-coordinate for centre of mass of AC = $\dfrac{1}{2}CX = \dfrac{\sqrt{5}}{2}$

y-coordinate for centre of mass of BC = $CX + \dfrac{1}{2}(DY - CX) = \sqrt{5} + \dfrac{1}{2}\left(\dfrac{8\sqrt{5}}{5} - \sqrt{5}\right) = \dfrac{13\sqrt{5}}{10}$

Challenge continued

$$12\bar{y} = 4\left(\frac{4\sqrt{5}}{5}\right) + 5\left(\frac{\sqrt{5}}{2}\right) + 3\left(\frac{13\sqrt{5}}{10}\right)$$
$$= \frac{32\sqrt{5}}{10} + \frac{25\sqrt{5}}{10} + \frac{39\sqrt{5}}{10}$$
$$= \frac{96\sqrt{5}}{10}$$
$$\bar{y} = \frac{4\sqrt{5}}{5}$$

Find distance between C and the centre of mass:

$$\sqrt{5} - \frac{4\sqrt{5}}{5} = \frac{\sqrt{5}}{5} \text{ cm}$$

So distance between the centre of mass and C is $\frac{\sqrt{5}}{5}$ cm