Centres of mass of plane figures 2B

1 $m\begin{pmatrix}1\\-3\end{pmatrix}+m\begin{pmatrix}5\\7\end{pmatrix}=(m+m)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$\begin{pmatrix}m\\-3m\end{pmatrix}+\begin{pmatrix}5m\\7m\end{pmatrix}=2m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Simplify by dividing both sides by m.

$\begin{pmatrix}6m\\4m\end{pmatrix}=2m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

$\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

Centre of mass is (3, 2). — This is the midpoint of the line joining the two points.

2 $m\begin{pmatrix}2\\0\end{pmatrix}+m\begin{pmatrix}-1\\3\end{pmatrix}+m\begin{pmatrix}2\\-4\end{pmatrix}+m\begin{pmatrix}-1\\-2\end{pmatrix}=4m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$ as before.

$\begin{pmatrix}2\\-3\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Divide both sides by m.

$\begin{pmatrix}\frac{1}{2}\\-\frac{3}{4}\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Solve.

Centre of mass is $\left(\frac{1}{2},-\frac{3}{4}\right)$

3 $10\begin{pmatrix}2\\3\end{pmatrix}+15\begin{pmatrix}4\\2\end{pmatrix}+5\begin{pmatrix}6\\6\end{pmatrix}=50\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

$\begin{pmatrix}4\\6\end{pmatrix}+\begin{pmatrix}12\\6\end{pmatrix}+\begin{pmatrix}30\\30\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Divide both sides by 5.

$\begin{pmatrix}46\\42\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Simplify.

$\begin{pmatrix}4.6\\4.2\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$ — Solve.

Centre of mass is (4.6, 4.2).

4 $0.5\begin{pmatrix}6\\-3\end{pmatrix}+1.5\begin{pmatrix}2\\5\end{pmatrix}+2\begin{pmatrix}3\\2\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

4 is the total mass.

It's easier to use column vectors.

$$\begin{pmatrix}3\\-1.5\end{pmatrix}+\begin{pmatrix}3\\7.5\end{pmatrix}+\begin{pmatrix}6\\4\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}12\\10\end{pmatrix}=4\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}3\\2.5\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

You could leave your answer as a column vector.

The position vector is $(3\mathbf{i}+2.5\mathbf{j})$.

5 $m\begin{pmatrix}-1\\-1\end{pmatrix}+2m\begin{pmatrix}3\\2\end{pmatrix}+5m\begin{pmatrix}4\\-2\end{pmatrix}+2m\begin{pmatrix}-2\\5\end{pmatrix}=10m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

$$\begin{pmatrix}-1\\-1\end{pmatrix}+\begin{pmatrix}6\\4\end{pmatrix}+\begin{pmatrix}20\\-10\end{pmatrix}+\begin{pmatrix}-4\\10\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Divide both sides by m.

$$\begin{pmatrix}21\\3\end{pmatrix}=10\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

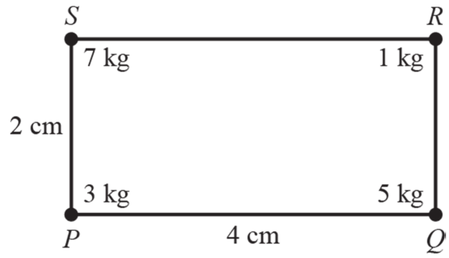
Simplify.

$$\begin{pmatrix}2.1\\0.3\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Solve.

Centre of mass is at (2.1, 0.3).

6



Draw a diagram. (Note that the plate is light.)

Draw the rectangle with the 2 'axes' (*PQ* and *PS*) in the bottom L.H. corner.

Taking *P* as the origin, and axes, *PQ* and *PS*, *P* is (0, 0); *Q* is (4, 0); *R* is (4, 2); *S* is (0, 2).

$$3\begin{pmatrix}0\\0\end{pmatrix}+5\begin{pmatrix}4\\0\end{pmatrix}+1\begin{pmatrix}4\\2\end{pmatrix}+7\begin{pmatrix}0\\2\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Total mass is 16 kg.

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}20\\0\end{pmatrix}+\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}0\\14\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}24\\16\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}1.5\\1\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Solve.

a Distance form *PQ* is 1 $(\bar{y})$.

b Distance from *PS* is 1.5 $(\bar{x})$.

7 $1\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+3\begin{pmatrix}p\\q\end{pmatrix}=(1+2+3)\begin{pmatrix}2\\0\end{pmatrix}$ — Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}8\\6\end{pmatrix}+\begin{pmatrix}3p\\3q\end{pmatrix}=\begin{pmatrix}12\\0\end{pmatrix}$ — Simplify.

$\begin{pmatrix}9+3p\\6+3q\end{pmatrix}=\begin{pmatrix}12\\0\end{pmatrix}$ — Collect terms.

$9+3p=12$
$6+3q=0$ — Equate **i** and **j** components.

$\Rightarrow p=1$
$q=-2$ — Solve for p and q.

8 $3m\begin{pmatrix}-3\\-4\end{pmatrix}+4m\begin{pmatrix}0.5\\4\end{pmatrix}+5m\begin{pmatrix}0\\-5\end{pmatrix}+7m\begin{pmatrix}x\\y\end{pmatrix}=19m\begin{pmatrix}0\\0\end{pmatrix}$ — Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$ $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$\begin{pmatrix}-9\\-12\end{pmatrix}+\begin{pmatrix}2\\16\end{pmatrix}+\begin{pmatrix}0\\-25\end{pmatrix}+\begin{pmatrix}7x\\7y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ — Divide by m and simplify.

$\begin{pmatrix}-7+7x\\-21+7y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ — Collect terms.

$-7+7x=0$
$-21+7y=0$ — Equate **i** and **j** components.

$x=1$
$y=3$ — Solve for x and y.

Coordinates of particles are (1, 3). — State answer.

9 We fix the origin and coordinate axes so that $A=(0,0)$, $B=(8,0)$, $C=(8,6)$ and $D=(0,6)$, now the total mass of the system is $M=300\text{g}+200\text{g}+600\text{g}+100\text{g}=1200\text{g}$

Let the centre of mass have coordinates (x, y) then the distance of the centre of mass from AB is y and the distance from AD is x then we have

$$300\begin{pmatrix}4\\0\end{pmatrix}+200\begin{pmatrix}8\\3\end{pmatrix}+600\begin{pmatrix}4\\6\end{pmatrix}+100\begin{pmatrix}0\\3\end{pmatrix}=1200\begin{pmatrix}x\\y\end{pmatrix}$$

so

$$3\begin{pmatrix}4\\0\end{pmatrix}+2\begin{pmatrix}8\\3\end{pmatrix}+6\begin{pmatrix}4\\6\end{pmatrix}+\begin{pmatrix}0\\3\end{pmatrix}=12\begin{pmatrix}x\\y\end{pmatrix}$$

so

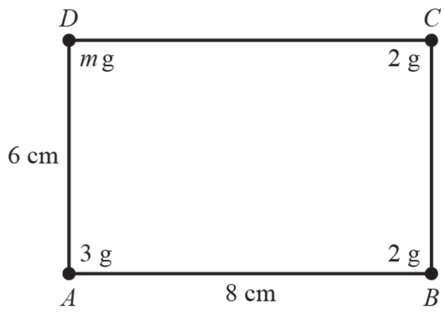
$$\begin{pmatrix}12+16+24\\6+36+3\end{pmatrix}=\begin{pmatrix}12x\\12y\end{pmatrix}$$

so

$$\begin{pmatrix}52\\45\end{pmatrix}=\begin{pmatrix}12x\\12y\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{13}{3}\\\frac{15}{4}\end{pmatrix}$$

10



Draw a diagram.

The card has no mass. Here 'g' is grams!

Let mass of particle of D be m g.

Taking axes through A, the coordinates of the particles are (0, 0), (8, 0), (8, 6) and (0, 6).

Here we have to set up our own axes.

$$3\begin{pmatrix}0\\0\end{pmatrix}+2\begin{pmatrix}8\\0\end{pmatrix}+2\begin{pmatrix}8\\6\end{pmatrix}+m\begin{pmatrix}0\\6\end{pmatrix}=(3+2+2+m)\begin{pmatrix}\bar{x}\\3\end{pmatrix}$$

'3 cm from A' means $\bar{y}=3$.

$$\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}16\\0\end{pmatrix}+\begin{pmatrix}16\\12\end{pmatrix}+\begin{pmatrix}0\\6m\end{pmatrix}=(7+m)\begin{pmatrix}\bar{x}\\3\end{pmatrix}$$

Simplify.

$$\begin{pmatrix}32\\12+6m\end{pmatrix}=\begin{pmatrix}(7+m)\bar{x}\\21+3m\end{pmatrix}$$

Collect terms.

$$12+6m=21+3m$$

Equate **j** components.

$$3m=9$$

$$m=3$$

Solve for m.

$$32=10\bar{x}$$

Equate **i** component and substitute for m.

$$3.2=\bar{x}$$

Solve for $\bar{x}$.

a $m=3$

b 3.2 cm

State your answer.

Challenge

By considering the triangle we can choose coordinates such that $A = (0,0)$, $B = (6,0)$ and $C = (3,4)$

Now the total mass of the system is $M = m + 0.7$, and the coordinates of the centre of mass satisfy

$$M\begin{pmatrix} x \\ y \end{pmatrix} = m\begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix} + 0.2\begin{pmatrix} \frac{9}{2} \\ 2 \end{pmatrix} + 0.5\begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

On the other hand we are given that the centre of mass is the centroid of the triangle. Since the triangle is isosceles by symmetry we must have that the intersection of the line connecting A and the midpoint of BC with the line connecting C and the midpoint of AB is the same as the intersection of the line connecting B and the midpoint of AC with the line connecting C and the midpoint of AB so we compute the first intersection, the line connecting C and the midpoint of AB can be written as

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the line connecting A and the midpoint of BC is given by

$$\mathbf{r} = s\begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

Hence at the intersection we have

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = s\begin{pmatrix} 9 \\ 4 \end{pmatrix}$$

And solving this gives the coordinates of the intersection as

$$\begin{pmatrix} 3 \\ \frac{4}{3} \end{pmatrix}$$

Going back to the equation for the centre of mass we have

$$(m + 0.7)\begin{pmatrix} 3 \\ \frac{4}{3} \end{pmatrix} = m\begin{pmatrix} \frac{3}{2} \\ 2 \end{pmatrix} + 0.2\begin{pmatrix} \frac{9}{2} \\ 2 \end{pmatrix} + 0.5\begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

And one can verify that $m = 0.2$ kg is the solution. This should be intuitive by the symmetry of the problem.