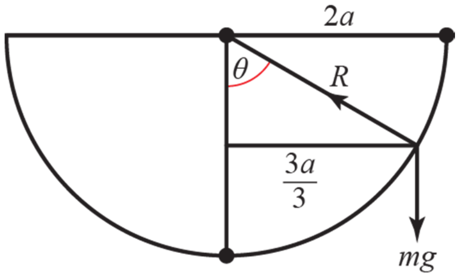
Mixed exercise 1

1



$R(\updownarrow) R\cos\theta = mg$

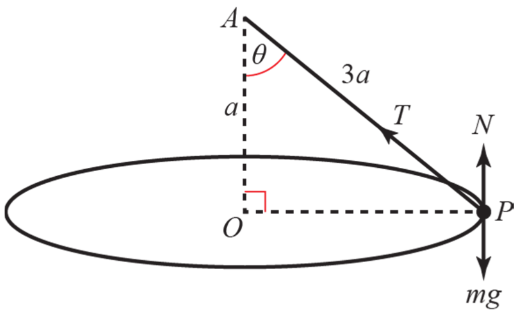
$R(\leftrightarrow) R\sin\theta = \frac{mv^2}{r} = \frac{2mu^2}{3a}$

Dividing $\Rightarrow \tan\theta = \frac{2u^2}{3ag}$, but

$\tan\theta = \frac{\frac{3a}{2}}{\frac{\sqrt{7}a}{2}} = \frac{3}{\sqrt{7}}$, so

$\frac{2u^2}{3ag} = \frac{3}{\sqrt{7}}$, $9ag = 2\sqrt{7}u^2$

2 a



N is the normal reaction of the table on P, T is the tension in the string, and θ is the angle between the string and the vertical. Right-angled triangle so

$OP = a\sqrt{8}$

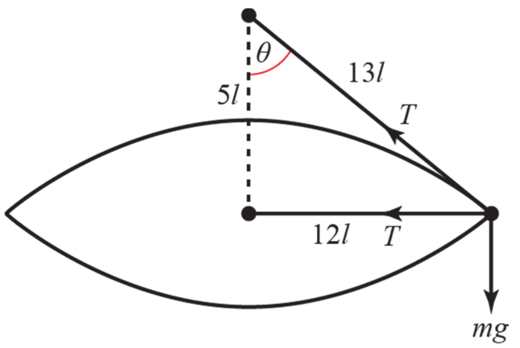
$R(\leftarrow): T\sin\theta = \frac{mv^2}{a\sqrt{8}}$

$T\frac{\sqrt{8}a}{3a} = \frac{m \times 4ga}{a\sqrt{8}}$

$\Rightarrow T = \frac{3mg}{2}$

b $R(\uparrow): T\cos\theta + N = mg \Rightarrow N = mg - \frac{3}{2}mg \times \frac{1}{3} = \frac{1}{2}mg$

3 a



Let θ be the angle between the string and the vertical.

We have a 5, 12, 13 triangle.

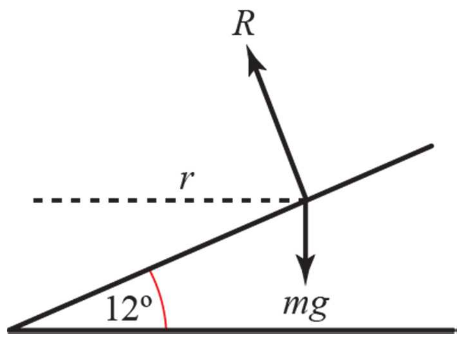
$R(\updownarrow) T\cos\theta = mg$

$T = \frac{mg}{\cos\theta} = \frac{13mg}{5}$

b $R(\leftrightarrow) T + T\sin\theta = \frac{mv^2}{r} \Rightarrow T\left(1 + \frac{12}{13}\right) = \frac{mv^2}{12l}, \frac{25}{13} \times \frac{13mg}{5} = 5mg = \frac{mv^2}{12l}$

$\Rightarrow v^2 = 60gl,\ v = \sqrt{60gl}$

4



R is the normal reaction of the surface on the car.

No friction.

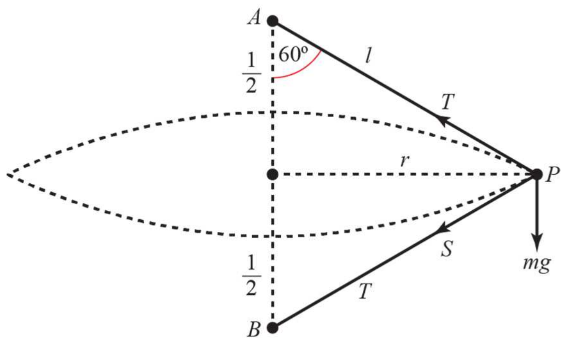
$$\text{R}(\updownarrow) R\cos 12^\circ = mg$$

$$\text{R}(\leftrightarrow) R\sin 12^\circ = \frac{mv^2}{r} = \frac{m\times 15^2}{r}$$

Dividing: $\tan 12^\circ = \dfrac{225}{gr}$

$$r = \frac{225}{g\tan 12^\circ} \approx 108\text{m}$$

5 a



T is the tension in AP and S is the tension in BP.
The triangle is equilateral (3 equal sides).

$$\text{R}(\updownarrow): T\cos 60^\circ = mg + S\cos 60^\circ$$
$$T - S = 2mg$$
$$\text{R}(\leftrightarrow): T\cos 30^\circ + S\cos 30^\circ = mr\omega^2$$
$$(T+S)\cos 30^\circ = ml\cos 30^\circ \times \omega^2$$
$$T + S = ml\omega^2$$

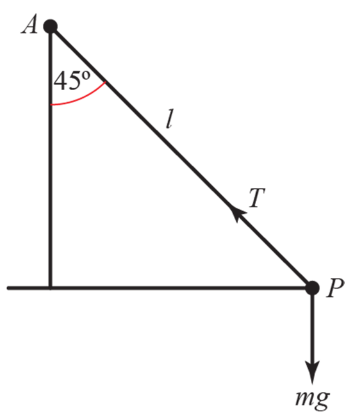
Adding these two equations gives

$$2T = 2mg + ml\omega^2,\ T = \frac{m}{2}(2g + l\omega^2).$$

b $S = T - 2mg = \dfrac{m}{2}(l\omega^2 - 2g)$

c Both strings taut $\Rightarrow l\omega^2 - 2g > 0,\ \omega^2 > \dfrac{2g}{l}$

6 a

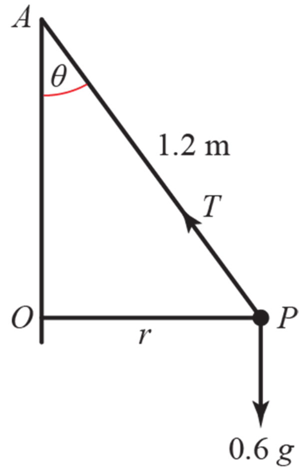


T is the tension in the string.

$\text{R}(\updownarrow): T\cos 45^\circ = mg,\ T = \sqrt{2}mg$

b $\text{R}(\leftrightarrow): T\cos 45^\circ = mr\omega^2 = ml\cos 45^\circ\omega^2,\ T = ml\omega^2,\ \omega = \sqrt{\dfrac{T}{ml}} = \sqrt{\dfrac{\sqrt{2}g}{l}}$

7 a



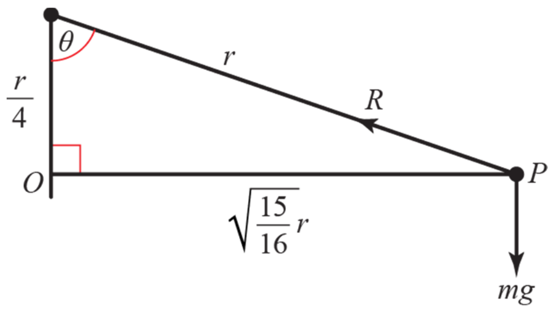
r is the radius of the circle,
T is the tension in the string and $\angle OAP$ is θ.

From the triangle, $r = 1.2\sin\theta$.

$$R(\leftrightarrow): T\sin\theta = mr\omega^2 = 0.6\times1.2\sin\theta\times9$$
$$T = 0.6\times1.2\times9 = 6.48\text{ N}$$

b $R(\updownarrow) T\cos\theta = mg,\ 6.48\cos\theta = 0.6g,\ \cos\theta = \frac{0.6g}{6.48} \approx 0.907,\ \theta \approx 25^\circ$

8 a



The angle between the radius through P and the vertical is θ.

P has angular speed $\omega\text{ rad s}^{-1}$

R is the reaction of the bowl on P.

$$R(\updownarrow): R\cos\theta = mg,\ R = 4mg.$$

b $R(\leftrightarrow): R\sin\theta = mr\omega^2 = m\times r\sin\theta\times\omega^2,\ \omega = \sqrt{\frac{4mg}{mr}} = \sqrt{\frac{4g}{r}}$

Three revolutions is 6π radians, time taken $= \frac{6\pi}{\sqrt{\frac{4g}{r}}} = 3\pi\sqrt{\frac{r}{g}}$

9 a
$$\frac{mv^2}{r} = \mu R = \mu mg$$
$$\frac{v^2}{rg} = \mu$$
$$\frac{21^2}{100\times9.8} = \mu$$
$$\mu = 0.45$$

b $\tan\alpha = \frac{35}{136}$

10 a $\frac{\sqrt{3}m}{4}\left(r\omega^2 + 2g\right)$

b Maximum speed gives the shortest time. At the maximum speed with the rod still on the surface of the sphere, $R = 0$.

Radius of the circle is $\frac{\sqrt{3}r}{2}$

When $R = 0$, $T\cos\alpha = mg$

$\Rightarrow T = \frac{mg}{\cos\alpha} = \frac{2mg}{\sqrt{3}}$

$T\sin\alpha = m \times \frac{\sqrt{3}r}{2} = \omega^2$

so $\frac{2mg}{\sqrt{3}} \times \frac{1}{2} = m \times \frac{\sqrt{3}r}{2} = \omega^2$

$\omega^2 = \frac{\sqrt{2g}}{3r}$

Time for one revolution $= \frac{2\pi}{\omega}$

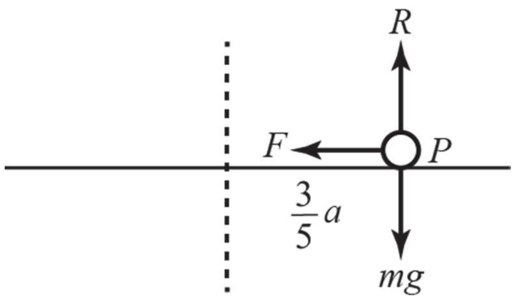
$= \pi\sqrt{\frac{4 \times 3r}{2g}}$

$= \pi\sqrt{\frac{6r}{g}}$

c i The minimum period decreases.

ii The minimum period increases.

11 a



F is the force due to friction,
R is the normal reaction.

$R(\updownarrow): R = mg$

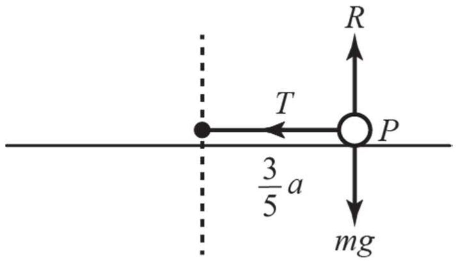
$R(\leftrightarrow): F = mr\omega^2$

If P is not to slip then

$\frac{3}{7}mg \geqslant m\frac{3}{5}a\omega^2$

$\therefore \omega^2 \leqslant \frac{5g}{7a}$

11 b



T is the tension in the elastic string.

$$T = \frac{\lambda x}{l} = \frac{\frac{5mg}{2}\times\left(\frac{3}{5}a - \frac{a}{2}\right)}{\frac{a}{2}} = \frac{5mg}{10} = \frac{mg}{2}$$

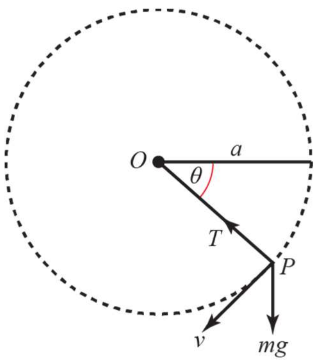
The limits for ω^2 depend on whether the friction is acting with the tension or against it.

$$\text{R}(\leftrightarrow): \frac{3}{7}mg + \frac{mg}{2} \geqslant m\frac{3}{5}a\omega^2, \omega^2 \leqslant \frac{5}{3a}\times\frac{13g}{14} = \frac{65g}{42a}$$

$$\text{or R}(\leftrightarrow): -\frac{3}{7}mg + \frac{mg}{2} \leqslant m\frac{3}{5}a\omega^2,\ \omega^2 \geqslant \frac{5}{3a}\times\frac{g}{14} = \frac{5g}{42a}$$

$$\frac{5g}{42a} \leqslant \omega^2 \leqslant \frac{65g}{42a}$$

12 a



Loss in P.E. = gain in K.E. so

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga\sin\theta$$

$$\Rightarrow v^2 = \frac{4}{3}ga + 2ga\sin\theta$$

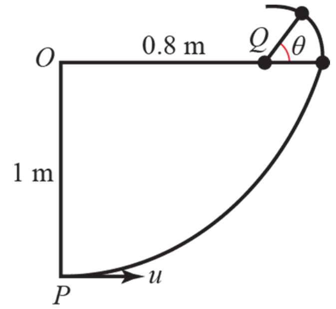
b Resolving towards O: $T - mg\sin\theta = \frac{mv^2}{a}$

$$T = \frac{4}{3}mg + 2mg\sin\theta + mg\sin\theta = mg\left(\frac{4}{3} + 3\sin\theta\right)$$

c $T = 0$ when $\sin\theta = -\frac{4}{9}$, $\theta = 206^\circ$

d When $v = 0$, $\sin\theta = -\frac{4}{6} = -\frac{2}{3}$, $(\theta \approx 222^\circ)$ so the particle can not complete the circle.

13



Consider the circle centre Q, radius 0.2 m.

When QP is at θ above the horizontal:

Energy: $\frac{1}{2}mw^2 + mg \times 0.2\sin\theta = \frac{1}{2}mv^2$,

$w^2 = v^2 - 0.4g\sin\theta$

where v is the speed when $\theta = 0$, and w the speed at angle θ.

Circular motion: $T + mg\sin\theta = \frac{mw^2}{r} = \frac{m(v^2 - 0.4g\sin\theta)}{0.2}$

$T = \frac{m(v^2 - 0.4g\sin\theta)}{0.2} - mg\sin\theta = \frac{m(v^2 - 0.6g\sin\theta)}{0.2} \geqslant 0$

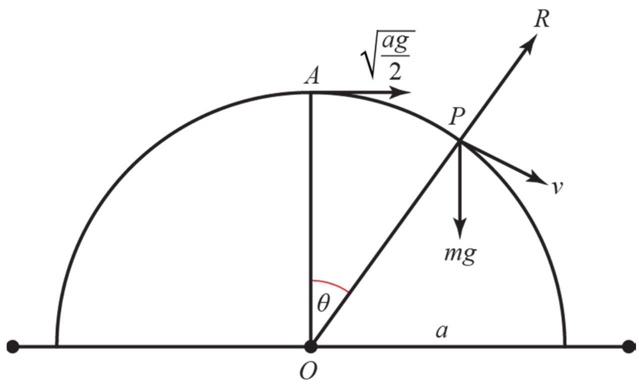
Looking at the larger circle, conservation of energy

$\Rightarrow \frac{1}{2}mv^2 + mg \times 1 = \frac{1}{2}mu^2$, $v^2 = u^2 - 2g$

At the top of the small circle, $\sin\theta = 1$,

$\Rightarrow u^2 - 2g - 0.6g \geqslant 0$, $u^2 \geqslant 2.6g$, $u \geqslant \sqrt{2.6g}$

14 a



R is the reaction between the particle and the surface.

If the level of P is the level of zero P.E., conservation of energy

$\Rightarrow \frac{1}{2}m\frac{ag}{2} + mga(1 - \cos\theta) = \frac{1}{2}mv^2$,

$v^2 = \frac{ga}{2} + 2ga(1 - \cos\theta)$

$= \frac{ga}{2}(5 - 4\cos\theta)$

b Resolving towards O: $mg\cos\theta - R = \frac{mv^2}{r} = \frac{mg}{2}(5 - 4\cos\theta)$

Substituting $\cos\theta = 0.9$: $R = mg \times 0.9 - \frac{mg}{2}(5 - 3.6) = 0.2mg > 0$

so P is still on the hemisphere.

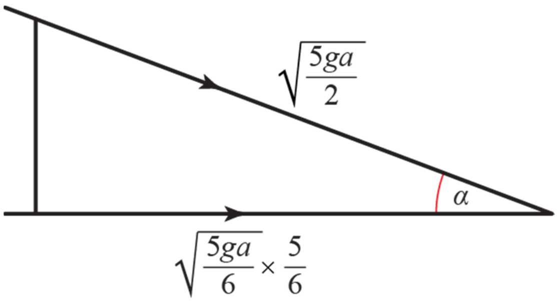
c **i** $R = 0 \Rightarrow \cos\theta = \frac{1}{2}(5 - 4\cos\theta)$, $3\cos\theta = \frac{5}{2}$, $\cos\theta = \frac{5}{6}$

ii $v^2 = \frac{ga}{2}(5 - 4\cos\theta) = \frac{ga}{2}\left(5 - \frac{10}{3}\right) = \frac{5ga}{6}$, $v = \sqrt{\frac{5ga}{6}}$

d By considering K.E. + P.E. at A and B, if v is the speed at B,

$\frac{1}{2}mv^2 = \frac{1}{2}m\frac{ag}{2} + mga$, $v^2 = \frac{5ga}{2}$, $v = \sqrt{\frac{5ga}{2}}$

14 e

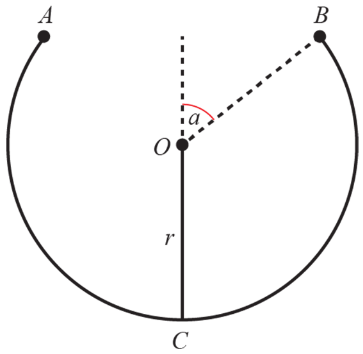


After the particle leaves the sphere the horizontal velocity remains constants $= \sqrt{\frac{5ga}{6}} \times \frac{5}{6}$

If α is the angle at which the particle strikes the table then $\cos\alpha = \frac{\sqrt{\frac{5ga}{6}} \times \frac{5}{6}}{\sqrt{\frac{5ga}{2}}} = \frac{5}{6\sqrt{3}}$

$\alpha \approx 61^\circ$

15 a



K.E. + P.E. at C = K.E. + P.E. at B.

If P.E. = 0 at C then

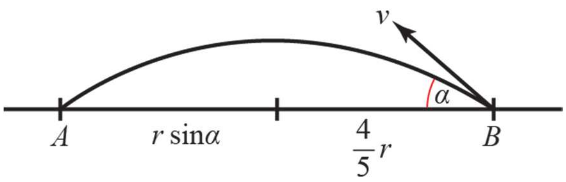
$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mg(r + r\cos\alpha) = \frac{1}{2}mv^2 + \frac{8}{5}mgr$$

$$v^2 = u^2 - \frac{16}{5}gr$$

b $u^2 = 4gr \Rightarrow v^2 = \frac{4}{5}gr$. Resolving towards O: $R + \frac{3}{5}mg = \frac{mv^2}{r} = \frac{4mg}{5}$, $R = \frac{mg}{5}$

c $R = 0$ at $B \Rightarrow \frac{3mg}{5} = \frac{mv^2}{r} = \frac{m\left(u^2 - \frac{16gr}{5}\right)}{r}$, $\frac{mu^2}{r} = \frac{3mg}{5} + \frac{16mg}{5}$, $u = \sqrt{\frac{19gr}{5}}$

d



The particle is now moving freely under gravity.
Horizontal distance

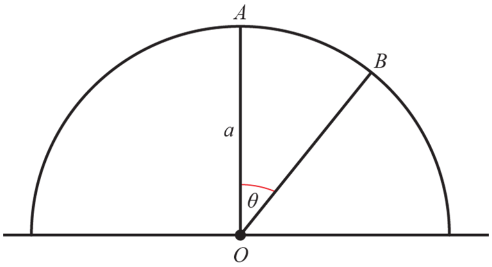
$= 2r\sin\alpha = \frac{8r}{5} = v\cos\alpha \times t$

so $t = \frac{8r}{3v}$

Vertical distance $= 0 = \frac{4v}{5}t - \frac{1}{2}gt^2 \Rightarrow t = \frac{8v}{5g} = \frac{8r}{3v}, \Rightarrow v = \sqrt{\frac{5rg}{3}}$

$\Rightarrow u^2 = \frac{5rg}{3} + \frac{16gr}{5} = \frac{73}{15}gr$; $u = \sqrt{\frac{73gr}{15}}$

16 a



Equating the K.E.+ P.E. at A and B:

$$\frac{1}{2}mu^2 + mga = \frac{1}{2}mv^2 + mga\cos\theta$$

$$\Rightarrow v^2 = u^2 + 2ga(1-\cos\theta)$$

Resolving towards O: $mg\cos\theta - R = \dfrac{mv^2}{a}$

$$R = 0 \Rightarrow ag\cos\theta = u^2 + 2ag(1-\cos\theta)$$

$$3ag\cos\theta = u^2 + 2ag$$

$$\cos\theta = \frac{u^2 + 2ag}{3ag}$$

b Conservation of energy from A to surface:

$$\frac{1}{2}mu^2 + mga = \frac{1}{2}m \times \frac{5ag}{2},\ u^2 = \frac{ag}{2},\ \cos\theta = \frac{5}{6},\ \theta \approx 34^\circ$$

Challenge

a At point

$$\left(x, x^2\right), \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\mathrm{R}(\uparrow): R\cos\theta = mg \quad (\mathbf{1})$$

$$\mathrm{R}(\rightarrow): R\sin\theta = mx\omega^2 \quad (\mathbf{2})$$

$$(\mathbf{2}) \div (\mathbf{1}): \tan\theta = \frac{x\omega^2}{g} \quad (\mathbf{3})$$

$$\tan\theta = \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\therefore 2x = \frac{x\omega^2}{g} \Rightarrow 2g = \omega^2$$

$$\Rightarrow \omega = \sqrt{2g}$$

Hence ω is independent of the vertical height.

Challenge

b From (**3**)

$\omega^2 = \frac{g\tan\theta}{x}$. For ω to be

independent of $x \Rightarrow \frac{g\tan\theta}{x} = k$ for constant k

$\Rightarrow \tan\theta = ax$ for constant a

$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = ax \Rightarrow y = \frac{1}{2}ax^2 + b$

Hence $\mathrm{f}(x) = px^2 + q$ for the constants p and q