Exam-style practice Paper 2

1 a Using partial fractions, we find that

$$\frac{1}{(r+2)(r+4)}=\frac{1}{2(r+2)}-\frac{1}{2(r+4)}.$$

So we can now use the method of differences to find

$$\sum_{r=1}^{n}\frac{1}{(r+2)(r+4)}=\sum_{r=1}^{n}\left(\frac{1}{2(r+2)}-\frac{1}{2(r+4)}\right)$$

$$=\frac{1}{6}-\frac{1}{10}+\frac{1}{8}-\frac{1}{12}+\frac{1}{10}-\frac{1}{14}+\cdots$$

$$+\frac{1}{2n}-\frac{1}{2(n+2)}+\frac{1}{2(n+1)}-\frac{1}{2(n+3)}$$

$$+\frac{1}{2(n+2)}-\frac{1}{2(n+4)}$$

$$=\frac{1}{6}+\frac{1}{8}-\frac{1}{2(n+3)}-\frac{1}{2(n+4)}$$

$$=\frac{n(7n+25)}{24(n+3)(n+4)}.$$

Note that most middling terms have cancelled with each other.

$p=7,$

$q=25.$

1 b For the base case, $n=1$,

$f(1)=2^3+3^3=35$ is divisible by 7.

We assume that the statement holds true for $n=k$. That is $f(k)=2^{k+2}+3^{2k+1}$ is divisible by 7.

Now for $n=k+1$,

$$f(k+1)=2^{k+3}+3^{2k+3}$$

$$=2\left(2^{k+2}\right)+3^2\left(3^{2k+1}\right)$$

$$=2\left(2^{k+2}\right)+9\left(3^{2k+1}\right)$$

$$=2\left(2^{k+2}\right)+2\left(3^{2k+1}\right)+7\left(3^{2k+1}\right)$$

$$=2f(k)+7\left(3^{2k+1}\right).$$

Since $f(k)$ is divisible by 7, $f(k+1)$ is also divisible by 7 and the statement holds for $n=k+1$.

The result is true for the base case $n=1$, and if it is true for $n=k$ then it is true for $n=k+1$. By mathematical induction, the result is true for all positive integers n.

2 a Since all coefficients of $f(x)$ are real, that means if there is a complex number as a root of the equation, there must also be the complex conjugate of that number as a root of the equation. In this case this means that since $1+4i$ is a root, $1-4i$ also is a root.

b $f(z)=z^4+az^3+30z^2+bz+85$

Since $f(1+4i)=0$ and $f(1-4i)=0$,

both $\left[z-(1+4i)\right]$ and $\left[z-(1-4i)\right]$

are factors of $f(z)$

Therefore

$$\left[z-(1+4i)\right]\left[z-(1-4i)\right]=\left[z^2-2z+17\right]$$

is also a factor of $f(z)$

We write

$$f(z)=\left[z^2-2z+17\right]\left[z^2+kz+5\right]$$

Equating coefficients of z^2 gives

$30=17+5-2k$

So $k=-4$

Therefore

$$f(z)=\left[z^2-2z+17\right]\left[z^2-4z+5\right]$$

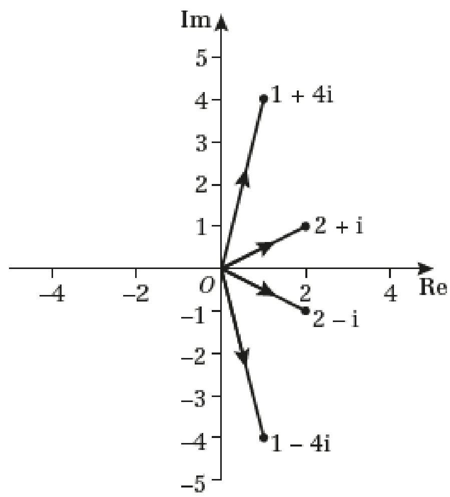
Solving $z^2-4z+5=0$ leads to

$$z=\frac{4\pm\sqrt{16-20}}{2}=2\pm i.$$

Thus we conclude that all the roots of the equation $f(x)$ are

$1+4i,\ 1-4i,\ 2+i$ and $2-i$.

c



3 First we need to find the point for which the tangent to the curve is perpendicular to the initial line. We form an expression for x and differentiate with respect to θ.

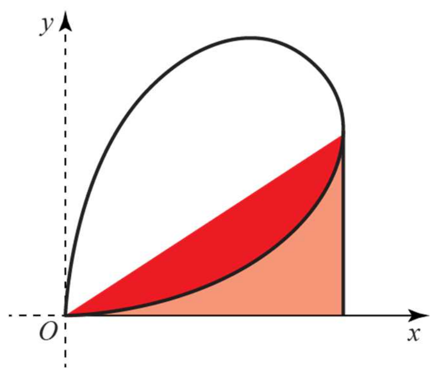
$$x = r\cos\theta$$
$$= 6\sin 2\theta\cos\theta$$
$$\frac{dx}{d\theta} = 12\cos 2\theta\cos\theta - 6\sin 2\theta\sin\theta$$
$$= 12(2\cos^2\theta - 1)\cos\theta - 12\cos\theta\sin^2\theta$$
$$= 36\cos^3\theta - 24\cos\theta$$
$$= 12\cos\theta(3\cos^2\theta - 2).$$

We now solve equal to 0 in order to find our required θ values. We choose to neglect the solutions arising from the $\cos\theta = 0$ factor, since a tangent at the origin is not what we are looking for even though it is perpendicular to the initial line.

So, $3\cos^2\theta - 2 = 0$ gives $\cos\theta = \pm\sqrt{\frac{2}{3}}$ and we choose to neglect the negative solution since $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Thus our tangent perpendicular to the initial line occurs at $\theta = \theta_A = \arccos\left(\sqrt{\frac{2}{3}}\right)$.

To find the area of the region, we will need to find the area of the sector that lies between $0 \leqslant \theta \leqslant \theta_A$ as shown in the diagram (red region).



$$A_{\text{sector}} = \frac{1}{2}\int_0^{\theta_A} (6\sin 2\theta)^2\,d\theta$$
$$= 18\int_0^{\theta_A} (\sin^2 2\theta)\,d\theta$$
$$= 9\int_0^{\theta_A} (1 - \cos 4\theta)\,d\theta$$
$$= 9\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\theta_A}$$
$$= 9\theta_A - \frac{9}{4}\sin 4\theta_A$$
$$= 9\arccos\left(\sqrt{\frac{2}{3}}\right) - \frac{9}{4}\sin\left(4\arccos\left(\sqrt{\frac{2}{3}}\right)\right)$$
$$\approx 4.13.$$

Now we find the area of the right-angle triangle bounded by the horizontal axis, the tangent and the line OA.

Using the formula

$$A_{\text{tri}} = \frac{1}{2} \times \text{Base} \times \text{Height}$$
$$= \frac{1}{2}|x||y|$$
$$= \frac{1}{2}r^2|\cos\theta||\sin\theta|$$
$$= 18(\sin^2 2\theta)|\cos\theta||\sin\theta|$$

and substituting in $\theta = \theta_A$, we find that

$$A_{\text{tri}} = \frac{16\sqrt{2}}{3}.$$

So, our shaded region is

$$A = A_{\text{tri}} - A_{\text{sector}}$$
$$= \frac{16\sqrt{2}}{3} - 4.125\ldots$$
$$\approx 3.42 \text{ units}^2 \text{(3sf)}.$$

4 a We evaluate the differentials at 0 until we have three non-zero terms.

$f(x)=\cos x\sinh 2x \Rightarrow f(0)=0$

$f'(x)=2\cos x\cosh 2x-\sin x\sinh 2x \Rightarrow f'(0)=2$

$f''(x)=3\cos x\sinh 2x-4\sin x\cosh 2x \Rightarrow f''(0)=0$

$f'''(x)=2\cos x\cosh x-11\sin x\sinh x \Rightarrow f'''(0)=2$

$f^{(4)}(x)=-7\cos x\sinh 2x-24\sin x\cosh 2x \Rightarrow f^{(4)}(0)=0$

$f^{(5)}(x)=-38\cos x\cosh x-41\sin x\sinh x \Rightarrow f^{(5)}(0)=-38.$

Now we use the standard Maclaurin series expansion and obtain

$$\cos x\sinh 2x \approx 2x+\frac{2x^3}{3!}+\frac{-38x^5}{5!}$$
$$=2x+\frac{x^3}{3}-\frac{19x^5}{60}.$$

b Using the approximation,

$$f(0.1)=\cos 0.1\sinh(2\times 0.1)$$
$$\approx 2\times 0.1+\frac{0.1^3}{3}-\frac{19\times 0.1^5}{60}$$
$$=\frac{1201981}{6000000}.$$

$$\text{error}=\left|\frac{\frac{1201981}{6000000}-\cos 0.1\sinh(2\times 0.1)}{\cos 0.1\sinh(2\times 0.1)}\right|\times 100$$

$$=2.754\times 10^{-6}\ \%(4\text{sf}).$$

5 Since we know that

$$\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\arctan\left(\frac{x}{a}\right)+C,$$ we can compute

$$\int_0^L\frac{1}{x^2+4}\,dx=\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^L$$
$$=\frac{1}{2}\arctan\left(\frac{L}{2}\right).$$

Now, since $\arctan\left(\frac{L}{2}\right)\to\frac{\pi}{2}$ as $L\to\infty$, we can conclude that the integral

$$\int_0^L\frac{1}{x^2+4}\,dx\to\frac{\pi}{4}\ \text{ as } L\to\infty.$$

6 The vertices $(0,2)$, $(k,0)$ and $(0,8)$ form a triangle with area $3k$.

The matrix $\begin{pmatrix}2 & 2\\ -3 & 5\end{pmatrix}$ increases the area by a factor of $(2\times 5)-(2\times -3)=16.$

So, $\text{Area}(T)=\frac{456}{16}=\frac{57}{2}.$

When we set this equal to the area of T, we obtain

$$\frac{1}{2}\times(8-2)\times k=\frac{57}{2}$$
$$3k=\frac{57}{2}$$
$$k=\frac{57}{6}=\frac{19}{2}$$
$$k=9.5.$$

7 a $$\int_{-1}^{1}\frac{1}{\sqrt{x^2+2x+2}}\,dx=\int_{-1}^{1}\frac{1}{\sqrt{(x+1)^2+1}}\,dx$$

Set $u=x+1$, $du=dx$ then the integral becomes $$\int_{-1}^{1}\frac{1}{\sqrt{(x+1)^2+1}}\,dx=\int_0^2\frac{1}{\sqrt{u^2+1}}\,du.$$

Now set

$v=\operatorname{arsinh} u$

$\Rightarrow u=\sinh v,$

$du=\cosh v\,dv=\sqrt{1+\sinh^2 v}\,dv=\sqrt{1+u^2}\,dv.$

Thus the integral now becomes

$$\int_0^2\frac{1}{\sqrt{u^2+1}}\,du=\int_0^{\operatorname{arsinh}2}\frac{1}{\sqrt{u^2+1}}\sqrt{u^2+1}\,dv$$
$$=\int_0^{\operatorname{arsinh}2}1\,dv$$
$$=[v]_0^{\operatorname{arsinh}2}$$
$$=\operatorname{arsinh}2.$$

Now in order to find the mean value of $f(x)$ over $[-1,1]$ we calculate

$$\frac{1}{1-(-1)}\operatorname{arsinh}2\approx 0.722\ (3\text{ d.p.})$$

7 b The mean value of $f(x)+2$ over $[-1,1]$ is

$$\frac{1}{1-(-1)}\int_{-1}^{1}\left(\frac{1}{\sqrt{x^2+2x+2}}+2\right)\mathrm{d}x$$

$$=\frac{1}{2}\int_{-1}^{1}\frac{1}{\sqrt{x^2+2x+2}}\,\mathrm{d}x+\frac{1}{2}\int_{-1}^{1}2\,\mathrm{d}x$$

$$\approx 0.722+\frac{1}{2}[2x]_{-1}^{1}$$

$$=0.722+\frac{4}{2}$$

$$=2.722 \text{ (3 d.p.)}$$

8 The point A has $x=4$ and so we find the y and z coordinates by solving

$$\frac{4-3}{-1}=\frac{y-2}{-2}=\frac{z-1}{3}.$$

$$\frac{y-2}{-2}=-1\Rightarrow y=4,$$

$$\frac{z-1}{3}=-1\Rightarrow z=-2.$$

So we have the coordinates $(4,4,-2)$ for the point A.

Next we find the perpendicular distance between A and Π.

Substituting $(4,4,-2)$ and the coefficients of $2x-y+3z-4=0$ into

$$\text{dist}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$ gives $\frac{3\sqrt{14}}{7}$

Thus there is a distance of $\frac{3\sqrt{14}}{7}$ between A and Π.

There will be the same distance between A' and Π.

This means the distance between A and A' is $\frac{6\sqrt{14}}{7}$.

9 a We look for a particular solution of the form

$$x=A+B\sin t+C\cos t,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=B\cos t-C\sin t,$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-B\sin t-C\cos t.$$

Now substituting these expressions into the differential equation:

$$-B\sin t-C\cos t$$
$$+2(B\cos t-C\sin t)$$
$$+3(A+B\sin t+C\cos t)$$
$$=21+15\cos t$$

which simplifies to

$$3A+2(B-C)\sin t+2(C+B)\cos t$$
$$=21+15\cos t$$

Comparing coefficients gives

$$A=7$$

$$B-C=0\Rightarrow B=C$$

$$2\times 2C=15\Rightarrow B=C=\frac{15}{4}.$$

Thus we have the particular solution

$$x=7+\frac{15}{4}(\sin t+\cos t).$$

9 b The auxiliary equation is $m^2+2m+3=0$,

with solutions $m=\dfrac{-2\pm\sqrt{4-12}}{2}$

$=-1\pm i\sqrt{2}.$

Thus we have the complementary function

$x_c=e^{-t}\left(D\cos\left(\sqrt{2}t\right)+E\sin\left(\sqrt{2}t\right)\right).$

We now add the complementary function and the particular integral we found in the previous part in order to obtain the general solution

$x_G=e^{-t}\left(D\cos\left(\sqrt{2}t\right)+E\sin\left(\sqrt{2}t\right)\right)+7+\frac{15}{4}(\sin t+\cos t).$

The first derivative of the general solution is

$$\frac{\mathrm{d}x_G}{\mathrm{d}t}=e^{-t}\left(-\sqrt{2}D\sin\left(\sqrt{2}t\right)+\sqrt{2}E\cos\left(\sqrt{2}t\right)\right)$$
$$-e^{-t}\left(D\cos\left(\sqrt{2}t\right)+E\sin\left(\sqrt{2}t\right)\right)$$
$$+\frac{15}{4}(\cos t-\sin t).$$

Now we use the initial conditions $x(0)=2,\ \frac{\mathrm{d}x}{\mathrm{d}t}(0)=3$ in order to find D and E.

$$x_G(0)=D+7+\frac{15}{4}=D+\frac{43}{4}=2$$

$$D=-\frac{35}{4}$$

$$\frac{\mathrm{d}x_G}{\mathrm{d}t}(0)=\sqrt{2}E-D+\frac{15}{4}=3$$

$$\sqrt{2}E=3-\frac{50}{4}$$

$$E=-\frac{19\sqrt{2}}{4}$$

Thus we have the solution

$$x=e^{-t}\left(-\frac{35}{4}\cos\left(\sqrt{2}t\right)-\frac{19\sqrt{2}}{4}\sin\left(\sqrt{2}t\right)\right)$$
$$+\frac{15}{4}(\sin t+\cos t)+7$$

c As $t\to\infty,\ x\to\frac{15}{4}(\sin t+\cos t)+7$ which is an oscillation about $x=7$.