Modelling with differential equations 8C

1 a $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+8x=0$

The auxiliary equation is

$m^2+4m+8=0$

$m=\frac{-4\pm\sqrt{16-32}}{2}=\frac{-4\pm\sqrt{-16}}{2}=-2\pm 2i$

So the general solution is $x=e^{-2t}(A\cos 2t+B\sin 2t)$

Using the initial conditions, when $t=0$, $x=2$ so $A\cos 0+B\sin 0=2\Rightarrow A=2$

$\dot{x}=-2e^{-2t}(A\cos 2t+B\sin 2t)+e^{-2t}(-2A\sin 2t+2B\cos 2t)$

When $t=0,\ \dot{x}=0\Rightarrow -2A+2B=0$

$\Rightarrow B=A=2$

Substituting for A and B gives the solution:

$x=e^{-2t}(2\cos 2t+2\sin 2t)=2e^{-2t}(\cos 2t+\sin 2t)$

b $x=2e^{-\frac{2\pi}{3}}\left(\cos\frac{2\pi}{3}+\sin\frac{2\pi}{3}\right)=2\times 0.1231\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)=0.0901$ (3 s.f.)

c As the auxiliary equation has complex roots, the motion is lightly damped.

2 $\frac{d^2x}{dt^2}+8\frac{dx}{dt}+12x=0$

The auxiliary equation is

$m^2+8m+12=0$

$(m+6)(m+2)=0$

$m=-6$ or -2

So the general solution is $x=Ae^{-6t}+Be^{-2t}$

When $t=0,\ x=4\Rightarrow A+B=4$ **(1)**

$\dot{x}=-6Ae^{-6t}-2Be^{-2t}$

$t=0,\ \dot{x}=0$ So $-6A-2B=0\Rightarrow 3A+B=0$ **(2)**

Subtracting equation (**1**) from equation (**2**) gives:

$2A=-4\Rightarrow A=-2$

Substituting in equation (**1**) gives $-2+B=4\Rightarrow B=6$

Substituting for A and B gives the solution:

$x=6e^{-2t}-2e^{-6t}$

3 a $\dfrac{d^2x}{dt^2}+2\dfrac{dx}{dt}+6x=0$

The auxiliary equation is

$m^2+2m+6=0$

$m=\dfrac{-2\pm\sqrt{4-24}}{2}=\dfrac{-2\pm\sqrt{-20}}{2}=-1\pm i\sqrt{5}$

So the general solution is $x=e^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)$

When $t=0$, $x=1$ so $A\cos 0+B\sin 0=1\Rightarrow A=1$

$\dot{x}=-e^{-t}(A\cos\sqrt{5}t+B\sin\sqrt{5}t)+e^{-t}(-\sqrt{5}A\sin\sqrt{5}t+\sqrt{5}B\cos\sqrt{5}t)$

When $t=0$, $\dot{x}=0\Rightarrow -A+\sqrt{5}B=0$

$\Rightarrow B=\dfrac{A}{\sqrt{5}}=\dfrac{1}{\sqrt{5}}$

Substituting for A and B gives the solution:

$$x=e^{-t}\left(\cos\sqrt{5}t+\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right)$$

b $\dot{x}=0,\quad t=T$

So $-e^{-T}\left(\cos\sqrt{5}T+\dfrac{1}{\sqrt{5}}\sin\sqrt{5}T\right)+e^{-T}\left(-\sqrt{5}\sin\sqrt{5}T+\dfrac{1}{\sqrt{5}}\sqrt{5}\cos\sqrt{5}T\right)=0 \qquad e^{-T}\neq 1$

$\Rightarrow -\cos\sqrt{5}T-\dfrac{1}{\sqrt{5}}\sin\sqrt{5}T-\sqrt{5}\sin\sqrt{5}T+\cos\sqrt{5}T=0$

$\Rightarrow \dfrac{5-1}{\sqrt{5}}\sin\sqrt{5}T=0\Rightarrow \sin\sqrt{5}T=0$

$\Rightarrow \sqrt{5}T=\pi, 2\pi, 3\pi,\ldots \qquad$ as $T>0$

So the smallest value of T for which P is at rest is $\dfrac{\pi}{\sqrt{5}}=1.40$ (3 s.f.)

4 a $\dfrac{d^2x}{dt^2}+4k\dfrac{dx}{dt}+4k^2x=0$

The auxiliary equation is

$m^2+4km+4k^2=0$

$(m+2k)(m+2k)=0$

$m=-2k$

So $x=(A+Bt)e^{-2kt}$

When $t=0$, $x=0$ so $Ae^0=0\Rightarrow A=0$

$\dot{x}=-2k(A+Bt)e^{-2kt}+Be^{-2kt}$

When $t=0$, $\dot{x}=u\Rightarrow -2kA+B=u$

As $A=0\Rightarrow B=u$

Substituting for A and B gives the solution:

$x=ute^{-2kt}$

4 b The particle comes to stop when $v = 0$, i.e. when

$$\frac{dx}{dt} = ue^{-2kt} - 2kute^{-2kt} = 0$$

$$\Rightarrow ue^{-2kt}(1-2kt) = 0$$

Assuming $u \neq 0$ this means $1-2kt = 0 \Rightarrow t = \frac{1}{2k}$

So in this model the particle will stop after $\frac{1}{2k}$ s

5 a The force of $6x$ N and the resistance force are acting in the same direction

So using $F = ma$ gives $-(6x+2v) = 2\ddot{x}$

Rearranging this equation gives

$$2\ddot{x} + 2v + 6x = 0$$

$$\Rightarrow \ddot{x} + \dot{x} + 3x = 0 \quad \text{as required (as } v = \dot{x}\text{)}$$

b Solving the auxiliary equation:

$$m^2 + m + 3 = 0$$

$$m = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{-1 \pm i\sqrt{11}}{2}$$

So $x = e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right)$

When $t = 0$, $x = 1$ so $e^0(A\cos 0 + B\sin 0) = 1 \Rightarrow A = 1$. Hence

$$\dot{x} = -\frac{1}{2}e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{11}}{2}t + B\sin\frac{\sqrt{11}}{2}t\right) + e^{-\frac{1}{2}t}\left(-\frac{\sqrt{11}}{2}A\sin\frac{\sqrt{11}}{2}t + \frac{\sqrt{11}}{2}B\cos\frac{\sqrt{11}}{2}t\right)$$

When $t = 0$, $\dot{x} = 2$ so $-\frac{1}{2}A + \frac{\sqrt{11}}{2}B = 2$

As $A = 1 \Rightarrow \frac{\sqrt{11}}{2}B = 2 + \frac{1}{2} = \frac{5}{2}$

$$\Rightarrow B = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}$$

Substituting for A and B gives the solution:

$$x = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t\right)$$

c When $t = 2$ we have:

$$x = e^{-\frac{1}{2}2}\left(\cos\frac{\sqrt{11}}{2}2 + \frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}2\right) = e^{-1}\left(\cos\sqrt{11} + \frac{5\sqrt{11}}{11}\sin\sqrt{11}\right) = -0.459\text{ m (3 s.f.)}$$

d As $\left(\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}\sin\frac{\sqrt{11}}{2}t\right)$ is bounded, the long-term behaviour of P is determined by $e^{-\frac{1}{2}t}$, which will decay to 0. So the maximum displacement will decrease exponentially.

6 Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2}+9x=0$

The auxiliary equation is

$m^2+9=0 \Rightarrow m=\pm 3i$

So the complementary function is $x = A\cos 3t + B\sin 3t$

The form of the particular integral is $x=\lambda\cos t+\mu\sin t$

$\frac{dx}{dt}=-\lambda\sin t+\mu\cos t$ and $\frac{d^2x}{dt^2}=-\lambda\cos t-\mu\sin t$

Substituting into $\frac{d^2x}{dt^2}+9x=k\cos t$ gives:

$-\lambda\cos t-\mu\sin t+9\lambda\cos t+9\mu\sin t=k\cos t$

Equating coefficients of $\cos t$: $8\lambda=k \Rightarrow \lambda=\frac{k}{8}$

Equating coefficients of $\sin t$: $8\mu=0 \Rightarrow \mu=0$

So a particular interval is $\frac{k}{8}\cos t$

The general solution is $x = A\cos 3t + B\sin 3t+\frac{k}{8}\cos t$

When $t=0$, $x=0$ so $A\cos 0+B\sin 0+\frac{k}{8}\cos 0=0 \Rightarrow A=-\frac{k}{8}$. Hence

$\dot{x}=-3A\sin 3t+3B\cos 3t-\frac{k}{8}\sin t$

When $t=0$, $\dot{x}=\frac{k}{5}$ so $-3A\sin 0+3B\cos 0-\frac{k}{8}\sin 0=\frac{k}{5} \Rightarrow B=\frac{k}{15}$

Substituting for A and B gives the solution:

$$x=-\frac{k}{8}\cos 3t+\frac{k}{15}\sin 3t+\frac{k}{8}\cos t$$

7 Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} + 5k\frac{dx}{dt} + 6k^2x = 0$

The auxiliary equation is

$$m^2 + 5km + 6k^2 = 0$$
$$(m + 2k)(m + 3k) = 0$$
$$m = -2k \text{ or } -3k$$

So the complementary function is $x = Ae^{-2kt} + Be^{-3kt}$

Try a constant for the particular integral, $x = \lambda$, so $\dot{x} = \ddot{x} = 0$

Substituting into $\frac{d^2x}{dt^2} + 5k\frac{dx}{dt} + 6k^2x = 5kU$ gives:

$$6k^2\lambda = 5kU \Rightarrow \lambda = \frac{5U}{6k}$$

So the general solution is $x = Ae^{-2kt} + Be^{-3kt} + \frac{5U}{6k}$

Now use the initial conditions to determine A and B

When $t = 0$, $x = 0$ so $A + B + \frac{5U}{6k} = 0 \Rightarrow A = -\frac{5U}{6k} - B$ **(1)**

$\dot{x} = -2kAe^{-2kt} - 3kBe^{-3kt}$

When $t = 0$, $\dot{x} = U$ so $-2kA - 3kB = U \Rightarrow 2kA = -3kB - U$ **(2)**

Substituting from equation (**1**) into equation (**2**) gives:

$$2kA = -3kB - U$$
$$\Rightarrow -\frac{5U}{3} - 2kB = -3kB - U$$
$$\Rightarrow B = \frac{2U}{3k}$$

So from equation (**1**) $A = -\frac{5U}{6k} - \frac{2U}{3k} = -\frac{9U}{6k} = -\frac{3U}{2k}$

Substituting for A and B gives the solution:

$$x = \frac{2U}{3k}e^{-3kt} - \frac{3U}{2k}e^{-2kt} + \frac{5U}{6k}$$

8 a Try a particular integral of the form $x = \lambda\cos t + \mu\sin t$

$\frac{dx}{dt} = -\lambda\sin t + \mu\cos t$ and $\frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substituting into $2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 100\cos t$ gives:

$-2\lambda\cos t - 2\mu\sin t - 3\lambda\sin t + 3\mu\cos t + \lambda\cos t + \mu\sin t = 100\cos t$

$\Rightarrow (3\mu - \lambda)\cos t - (3\lambda + \mu)\sin t = 100\cos t$

Equating coefficients of $\cos t$: $3\mu - \lambda = 100 \Rightarrow \lambda = 3\mu - 100$ **(1)**

Equating coefficients of $\sin t$: $3\lambda + \mu = 0 \Rightarrow \mu = -3\lambda$ **(2)**

Substituting equation (**2**) in equation (**1**) gives:

$\lambda = -9\lambda - 100 \Rightarrow \lambda = -10$

So $\mu = -3\lambda = 30$

So the particular solution is $x = 30\sin t - 10\cos t$ as required.

b To find the general solution solving the corresponding homogeneous equation $2\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0$

$2m^2 + 3m + 1 = 0$

$(2m+1)(m+1) = 0$

$m = -\frac{1}{2}$ or -1

Hence the general solution is $x = Ae^{-t} + Be^{-\frac{1}{2}t} + 30\sin t - 10\cos t$

c To find the value of x at 5 seconds, use initial conditions to find the values of A and B.

When $t = 0$, $x = 0$ so $Ae^0 + Be^0 + 30\sin 0 - 10\cos 0 = 0 \Rightarrow A + B = 10$ **(1)**

$\dot{x} = -Ae^{-t} - \frac{1}{2}Be^{-\frac{1}{2}t} + 30\cos t + 10\sin t$

When $t = 0$, $\dot{x} = 0$ so $-Ae^0 - \frac{1}{2}Be^{0t} + 30\cos 0 + 10\sin 0 = 0 \Rightarrow 2A + B = 60$ **(2)**

Subtracting equation (**1**) from equation (**2**) gives

$A = 60 - 10 = 50$, so from equation (**1**) $B = -40$

Substituting for A and B gives the solution:

$x = 50e^{-t} - 40e^{-\frac{1}{2}t} + 30\sin t - 10\cos t$

At $t = 5$

$x = 50e^{-5} - 40e^{-2.5} + 30\sin 5 - 10\cos 5 = -34.55$ m (2 d.p.)