Modelling with differential equations 8B

1 a The particle moves with simple harmonic motion.

b Rewriting $\frac{d^2x}{d^2t} = -9x$ as $\frac{d^2x}{d^2t} + 9x = 0$

The auxiliary equation is

$m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

So $x = A\cos 3t + B\sin 3t$

Using the initial conditions, when $t = 0, x = 2$ so $A\cos 0 + B\sin 0 = 2 \Rightarrow A = 2$

When $t = 0$, $v = \frac{dx}{dt} = 3$

$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t$

So $-3A\sin 0 + 3B\cos 0 = 3 \Rightarrow B = 1$

Substituting for A and B gives the solution:

$x = 2\cos 3t + \sin 3t$

c Writing the solution to part **b** in the form $R\sin(\theta + \alpha)$

$R\sin(3t + \alpha) = R\sin\alpha\cos 3t + R\cos\alpha\sin 3t$

So $R\sin\alpha = 2$ and $R\cos\alpha = 1$

$R^2\sin^2\alpha + R^2\cos^2\alpha = R^2(\sin^2\alpha + \cos^2\alpha) = R^2 = 5 \Rightarrow R = \sqrt{5}$

$\tan\alpha = 2 \Rightarrow \alpha = 1.12$ (3 s.f.)

So the equation can be written as $x = \sqrt{5}\sin(3t + 1.12)$

Hence $x_{max} = \sqrt{5}$

2 a Rewriting $\ddot{x} = -16x$ as $\ddot{x} + 16x = 0$ and solving the auxiliary equation

$m^2 + 16 = 0$

$\Rightarrow m = \pm 4i$

Hence $x = A\cos 4t + B\sin 4t$

Using the initial conditions, when $t = 0, x = 2$ so $A\cos 0 + B\sin 0 = 5 \Rightarrow A = 5$

When $t = 0$, $\dot{x} = 2$

$\dot{x} = -4A\sin 4t + 4B\cos 4t$

So $-4A\sin 0 + 4B\cos 0 = 2 \Rightarrow B = \frac{1}{2}$

Substituting for A and B gives the solution:

$x = 5\cos 4t + \frac{1}{2}\sin 4t$

2 b The period of motion can be found by solving $4t = 2\pi$

So the period of motion is $\frac{\pi}{2}$

To find the maximum displacement, write x in the form $R\sin(\theta+\alpha)$:

$R\sin(4t+\alpha) = R\sin\alpha\cos 4t + R\cos\alpha\sin 4t$

So $R\sin\alpha = 5$ and $R\cos\alpha = \frac{1}{2}$

$R^2 = 25 + \frac{1}{4} \Rightarrow R = \sqrt{\frac{101}{4}} = \frac{\sqrt{101}}{2}$

$\tan\alpha = 10 \Rightarrow \alpha = 1.47$ (3 s.f.)

So the equation can be written as $x = \frac{\sqrt{101}}{2}\sin(4t+1.47)$

Hence $x_{max} = \frac{\sqrt{101}}{2}$

3 a The particle moves with simple harmonic motion.

b The acceleration is proportional to the distance from the origin, so $\ddot{x} = ax$ for some constant of proportionality, a.

When $x = 1$, $\ddot{x} = -5$, this gives $-5 = a$

So $\ddot{x} = -5x$

c Rewriting $\ddot{x} = -5x$ as $\ddot{x} + 5x = 0$ and solving the auxiliary equation

$m^2 + 5 = 0$

$\Rightarrow m = \pm\sqrt{5}\text{i}$

Hence $x = A\cos\sqrt{5}t + B\sin\sqrt{5}t$

Using the initial conditions, when $t = 0$ and $x = 5$ so $A\cos 0 + B\sin 0 = 5 \Rightarrow A = 5$

When $t = 0$, $\dot{x} = 6$

$\dot{x} = -\sqrt{5}A\sin\sqrt{5}t + \sqrt{5}B\cos\sqrt{5}t$

So $-\sqrt{5}A\sin 0 + \sqrt{5}B\cos 0 = 6 \Rightarrow B = \frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5}$

Substituting for A and B gives the solution:

$$x = 5\cos\sqrt{5}t + \frac{6\sqrt{5}}{5}\sin\sqrt{5}t$$

d Rewriting x in the form $R\sin(\theta+\alpha)$

$R^2 = 5^2 + \left(\frac{6\sqrt{5}}{5}\right)^2 = 25 + \frac{180}{25} = \frac{625}{25} + \frac{180}{25} = \frac{805}{25} \Rightarrow R = \frac{\sqrt{805}}{5}$

$\tan\alpha = \frac{5}{\frac{6\sqrt{5}}{5}} = \frac{25}{6\sqrt{5}} \Rightarrow \alpha = 1.08$ (3 s.f.)

So the equation can be written as $x = \frac{\sqrt{805}}{5}\sin(\sqrt{5}t + 1.08)$, hence $x_{max} = \frac{\sqrt{805}}{5}$

4 a $\dfrac{d^2x}{dt^2} = -kx$

As $\dfrac{d^2x}{d^2t} = -7$ when $x = 2$, this gives $-7 = -2k \Rightarrow k = \dfrac{7}{2}$

b Rewriting $\dfrac{d^2x}{d^2t} = -kx = -\dfrac{7}{2}x$ as $\dfrac{d^2x}{d^2t} + \dfrac{7}{2}x = 0$

The auxiliary equation is

$m^2 + \dfrac{7}{2} = 0$

$\Rightarrow m = \pm\sqrt{\dfrac{7}{2}}\mathrm{i}$

So $x = A\cos\sqrt{\dfrac{7}{2}}t + B\sin\sqrt{\dfrac{7}{2}}t$

Using the initial conditions, when $t = 0, x = 6$ so $A\cos 0 + B\sin 0 = 6 \Rightarrow A = 6$

When $t = 0$, $\dot{x} = \dfrac{dx}{dt} = 1$

$\dfrac{dx}{dt} = -\sqrt{\dfrac{7}{2}}A\sin\sqrt{\dfrac{7}{2}}t + \sqrt{\dfrac{7}{2}}B\cos\sqrt{\dfrac{7}{2}}t$

So $-\sqrt{\dfrac{7}{2}}A\sin 0 + \sqrt{\dfrac{7}{2}}B\cos 0 = 1 \Rightarrow B = \sqrt{\dfrac{2}{7}}$

Substituting for A and B gives the solution:

$$x = 6\cos\sqrt{\frac{7}{2}}t + \sqrt{\frac{2}{7}}\sin\sqrt{\frac{7}{2}}t$$

c The period of motion can be found by solving $\sqrt{\dfrac{7}{2}}t = 2\pi$

So the period of motion is $\dfrac{2\sqrt{2}\pi}{\sqrt{7}} = 3.36$ seconds (2 d.p.)

5 a Rewriting the equation as $\frac{d^2x}{dt^2} + 2.25x = 0$ and solving the auxiliary equation:

$m^2 + 2.25 = 0$

$\Rightarrow m = \pm 1.5\text{i}$

So $x = A\sin 1.5t + B\cos 1.5t$

At $t = 2$, $x = 1.3$, the maximum displacement so

$A\sin 3 + B\cos 3 = 1.3$ **(1)**

At maximum displacement, the velocity of the boat is 0, so at $t = 2$, $v = 0$, so

$\dot{x} = 1.5A\cos 1.5t - 1.5B\sin 1.5t$

So $1.5A\cos 3 - 1.5B\sin 3 = 0$

$\Rightarrow A\cos 3 = B\sin 3 \Rightarrow A = B\tan 3$

Substituting for A into equation (1) gives:

$B\tan 3\sin 3 + B\cos 3 = 1.3$

$B(-0.1425 \times .1411 + -.9900) = 1.3$

$B = -1.287$ (3 d.p.)

So $A = B\tan 3 = -1.287 \times \tan 3 = 0.183$ (3 d.p.)

Substituting for A and B gives the solution:

$x = 0.183\sin 1.5t - 1.287\cos 1.5t$

b The time elapsed between the boat being at its highest and its lowest point is half of the full period. The period of motion is found by solving:

$1.5t = 2\pi \Rightarrow t = \frac{4\pi}{3}$

Hence the time elapsed between the boat being at its highest and lowest point is $\frac{2\pi}{3}$ seconds

c The model assumes that the amplitude of the boat's motion is fixed and that it will float to a fixed highest and fixed lowest point. This does not account for any changes in the motion of the sea over time due to tides, weather, etc.

6 a The particle moves with a simple harmonic motion.

b Rewriting the equation as $\ddot{x} + 200x = 0$ and solving the auxiliary equation

$m^2 + 200 = 0$

$\Rightarrow m = \pm\text{i}\sqrt{200} = \pm 10\text{i}\sqrt{2}$

So $x = A\cos 10\sqrt{2}t + B\sin 10\sqrt{2}t$

Using the initial conditions, when $t = 0$, $x = 0.3$ so $A\cos 0 + B\sin 0 = 0.3 \Rightarrow A = 0.3$

When $t = 0$, $\dot{x} = 0$

$\dot{x} = -10\sqrt{2}A\sin 10\sqrt{2}t + 10\sqrt{2}B\cos 10\sqrt{2}t$

So $-10\sqrt{2}A\sin 0 + 10\sqrt{2}B\cos 0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = 0.3\cos 10\sqrt{2}t$

6 c The period of motion can be found by solving $10\sqrt{2}t = 2\pi$

So the period of motion is $\frac{2\pi}{10\sqrt{2}} = \frac{\sqrt{2}\pi}{10}$ seconds

The largest displacement occurs when $\cos 10\sqrt{2}t = 1$ and hence $x = 0.3$, so the amplitude is 0.3 m.

d The velocity is given by $\dot{x} = -3\sqrt{2}\sin 10\sqrt{2}t$

The maximum velocity occurs when $\sin 10\sqrt{2}t = -1$ and so $v_{max} = 3\sqrt{2}\,\text{m s}^{-1}$

7 a Rewriting the equation as $\ddot{x} + \frac{100}{0.64}x = 0$ and solving the auxiliary equation

$m^2 + \frac{100}{0.84} = 0$

$\Rightarrow m = \pm\frac{10}{0.8}\text{i} = \pm 12.5\text{i}$

So $x = A\cos 12.5t + B\sin 12.5t$

Using the initial conditions, when $t = 0$, $x = 1$ so $A\cos 0 + B\sin 0 = 1 \Rightarrow A = 1$

When $t = 0$, $\dot{x} = 0$

$\dot{x} = -12.5A\sin 12.5t + 12.5B\cos 12.5t$

So $-12.5A\sin 0 + 12.5B\cos 0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = \cos 12.5t$

b The period of motion can be found by solving $12.5t = 2\pi$

So the period of motion is $\frac{2\pi}{12.5} = \frac{4\pi}{25}$ seconds

8 a Rewriting the equation as $\ddot{x} + 320x = 0$ and solving the auxiliary equation

$m^2 + 320 = 0$

$\Rightarrow m = \pm\sqrt{320}\text{i} = \pm\sqrt{64\times 5}\text{i} = \pm 8\sqrt{5}\text{i}$

So $x = A\cos 8\sqrt{5}t + B\sin 8\sqrt{5}t$

Using the initial conditions, when $t = 0$, $x = 8$ so $A\cos 0 + B\sin 0 = 8 \Rightarrow A = 8$

When $t = 0$, $\dot{x} = 0$

$v = \dot{x} = -8\sqrt{5}A\sin 8\sqrt{5}t + 8\sqrt{5}B\cos 8\sqrt{5}t$

So $-8\sqrt{5}A\sin 0 + 8\sqrt{5}B\cos 0 = 0 \Rightarrow B = 0$

Substituting for A and B gives the solution:

$x = 8\cos 8\sqrt{5}t$

b The period of the resulting oscillations can be found by solving $8\sqrt{5}t = 2\pi$

So the period of oscillation is $\frac{2\pi}{8\sqrt{5}} = \frac{2\sqrt{5}\pi}{40} = 0.351$ seconds (3 d.p.)

9 a Rewriting the equation as $\ddot{x}+\frac{250}{3}x=0$ and solving the auxiliary equation

$m^2+\frac{250}{3}=0$

$\Rightarrow m=\pm\frac{5\sqrt{10}}{\sqrt{3}}\text{i}$

So $x=A\cos\frac{5\sqrt{10}}{\sqrt{3}}t+B\sin\frac{5\sqrt{10}}{\sqrt{3}}t$

Using the initial conditions, when $t=0$, $x=15$ so $A\cos 0+B\sin 0=15\Rightarrow A=15$

When $t=0$, $\dot{x}=0$

$v=\dot{x}=-\frac{5\sqrt{10}}{\sqrt{3}}A\sin\frac{5\sqrt{10}}{\sqrt{3}}t+\frac{5\sqrt{10}}{\sqrt{3}}B\cos\frac{5\sqrt{10}}{\sqrt{3}}t$

So $-\frac{5\sqrt{10}}{\sqrt{3}}A\sin 0+\frac{5\sqrt{10}}{\sqrt{3}}B\cos 0=0\Rightarrow B=0$

Substituting for A and B gives the solution:

$x=15\cos\frac{5\sqrt{10}}{\sqrt{3}}t$

b The period of motion can be found by solving $\frac{5\sqrt{10}}{\sqrt{3}}t=2\pi$

So the period of motion is $\frac{2\sqrt{3}\pi}{5\sqrt{10}}=\frac{2\sqrt{30}\pi}{50}=\frac{\sqrt{30}\pi}{25}=0.688$ seconds (3 d.p.)

The maximum displacement occurs when $\cos\frac{5\sqrt{10}}{\sqrt{3}}t=1$, so $x_{max}=15$

Hence the amplitude is 15 cm

c The model does not account for air resistance, which will cause the pendulum to stop swinging eventually. A refinement to the model would be to incorporate a damping effect so that as $t\to\infty, x\to 0$.