Methods in differential equations 7D

1 a The auxiliary equation for the corresponding homogeneous equation is

$m^2+5m+6=0$

$(m+2)(m+3)=0$

$m=-2 \text{ or } -3$

So the complementary function is $y=Ae^{-2x}+Be^{-3x}$

A particular integral for the equation is of the form $y=\lambda e^x$, so $\frac{dy}{dx}=\lambda e^x$ and $\frac{d^2y}{dx^2}=\lambda e^x$

Substituting into $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=12e^x$ gives $12\lambda e^x=12e^x \Rightarrow \lambda=1$

So a particular integral is e^x

The general solution is $y=Ae^{-2x}+Be^{-3x}+e^x$

b When $x=0,\ y=1$ so $A+B+1=1 \Rightarrow A=-B$ **(1)**

$\frac{dy}{dx}=-3Ae^{-3x}-2Be^{-2x}+e^x$

When $x=0, \frac{dy}{dx}=0$ so $-3A-2B+1=0$

$\Rightarrow 3A+2B=1$ **(2)**

Substituting the value for A from equation (**1**) into equation (**2**)

$-3B+2B=1 \Rightarrow B=-1$

So $A=-B=1$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y=e^{-3x}-e^{-2x}+e^x$

2 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

The auxiliary equation is

$m^2+2m=0$

$m(m+2)=0$

$m=0 \text{ or } -2$

So the complementary function is $y=Ae^{0x}+Be^{-2x}=A+Be^{-2x}$

The form of the particular integral is $y=\lambda e^{2x}$, so $\frac{dy}{dx}=2\lambda e^{2x}$ and $\frac{d^2y}{dx^2}=4\lambda e^{2x}$

Substituting into $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=12e^{2x}$ gives: $4\lambda e^{2x}+4\lambda e^{2x}=12e^{2x} \Rightarrow \lambda=\frac{3}{2}$

So a particular integral is $\frac{3}{2}e^{2x}$

The general solution is $y=A+Be^{-2x}+\frac{3}{2}e^{2x}$

2 b When $x=0$, $y=2$ so $A+B+\frac{3}{2}=2 \Rightarrow A+B=\frac{1}{2}$ **(1)**

$\frac{\mathrm{d}y}{\mathrm{d}x}=-2B\mathrm{e}^{-2x}+3\mathrm{e}^{2x}$

When $x=0, \frac{\mathrm{d}y}{\mathrm{d}x}=6$ so $-2B+3=6 \Rightarrow B=-\frac{3}{2}$

Substituting into equation (**1**) $A-\frac{3}{2}=\frac{1}{2} \Rightarrow A=2$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y=2-\frac{3}{2}\mathrm{e}^{-2x}+\frac{3}{2}\mathrm{e}^{2x}$

3 Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-42y=0$

The auxiliary equation is

$m^2-m-42=0$

$(m-7)(m+6)=0$

$m=-6$ or 7

So the complementary function is $y=A\mathrm{e}^{-6x}+B\mathrm{e}^{7x}$

The form of the particular integral is $y=\lambda$, so $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$

Substituting in $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{\mathrm{d}y}{\mathrm{d}x}-42y=14$ gives: $-42\lambda=14 \Rightarrow \lambda=-\frac{1}{3}$

So the general solution is $y=A\mathrm{e}^{-6x}+B\mathrm{e}^{7x}-\frac{1}{3}$

Now applying the boundary conditions

When $x=0$, $y=0$ so $A+B-\frac{1}{3}=0 \Rightarrow 3A+3B=1$ **(1)**

$\frac{\mathrm{d}y}{\mathrm{d}x}=-6A\mathrm{e}^{-6x}+7B\mathrm{e}^{7x}$

When $x=0, \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{6}$ so $-6A+7B=\frac{1}{6}$ **(2)**

Adding 2 × equation (**1**) to equation (**2**) gives:

$6A+6B-6A+7B=2+\frac{1}{6}$

$\Rightarrow 13B=\frac{13}{6} \Rightarrow B=\frac{1}{6}$

Substituting into equation (**1**) $A+\frac{1}{6}=\frac{1}{3} \Rightarrow A=\frac{1}{6}$

Substituting values for A and B into the general solution gives the particular solution

$y=\frac{1}{6}\mathrm{e}^{-6x}+\frac{1}{6}\mathrm{e}^{7x}-\frac{1}{3}$

4 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+9y=0$

The auxiliary equation is $m^2+9=0 \Rightarrow m=\pm 3i$

So the complementary function is $y=A\cos 3x+B\sin 3x$

To find a particular integral consider functions of the form $y=\lambda\cos x+\mu\sin x$

$\frac{dy}{dx}=-\lambda\sin x+\mu\cos x$ and $\frac{d^2y}{dx^2}=-\lambda\cos x-\mu\sin x$

Substituting in $\frac{d^2y}{dx^2}+9y=16\sin x$ gives:

$-\lambda\cos x-\mu\sin x+9(\lambda\cos x+\mu\sin x)=16\sin x$

$\Rightarrow 8\lambda\cos x+8\mu\sin x=16\sin x$

$\Rightarrow \lambda=0, \mu=2$

So a particular integral is $2\sin x$

The general solution of the equation is $y=A\cos 3x+B\sin 3x+2\sin x$

b When $x=0, y=1$ so $A\cos 0+B\sin 0+2\sin 0=1 \Rightarrow A=1$

$\frac{dy}{dx}=-3A\sin 3x+3B\cos 3x+2\cos x$

When $x=0, \frac{dy}{dx}=8$ so $3B+2=8 \Rightarrow B=2$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$y=\cos 3x+2\sin 3x+2\sin x$

5 a Solving the corresponding homogeneous equation $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$

The auxiliary equation is $4m^2+4m+5=0$

$m=\frac{-4\pm\sqrt{16-80}}{8}=\frac{-4\pm\sqrt{-64}}{8}=\frac{-4\pm 8i}{8}=-\frac{1}{2}\pm i$ using the quadratic formula

So the complementary function is $y=e^{-\frac{1}{2}x}(A\cos x+B\sin x)$

The form of the particular integral is $y=\lambda\cos x+\mu\sin x$

$\frac{dy}{dx}=-\lambda\sin x+\mu\cos x$ and $\frac{d^2y}{dx^2}=-\lambda\cos x-\mu\sin x$

Substituting into $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=\sin x+4\cos x$ gives:

$-4\lambda\cos x-4\mu\sin x-4\lambda\sin x+4\mu\cos x+5\lambda\cos x+5\mu\sin x=\sin x+4\cos x$

Equating the coefficients of $\cos x$: $\lambda+4\mu=4$ **(1)**

Equating the coefficients of $\sin x$: $-4\lambda+\mu=1$ **(2)**

Adding 4 × equation (**1**) to equation (**2**) gives: $17\mu=17 \Rightarrow \mu=1$

And hence from equation (**1**) $\lambda=0$, so a particular integral is $\sin x$

The general solution is $y=e^{-\frac{1}{2}x}(A\cos x+B\sin x)+\sin x$

5 b When $x = 0, y = 0$ so $e^0 A = 0 \Rightarrow A = 0$

So the particular solution is $y = Be^{-\frac{1}{2}x}\sin x + \sin x$

$$\frac{dy}{dx} = Be^{-\frac{1}{2}x}\cos x - \frac{1}{2}Be^{-\frac{1}{2}x}\sin x + \cos x$$

When $x = 0, \frac{dy}{dx} = 0$ so $B + 1 = 0 \Rightarrow B = -1$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$$y = -e^{-\frac{1}{2}x}\sin x + \sin x = \sin x(1 - e^{-\frac{1}{2}x})$$

6 a Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$
$$m = 1 \text{ or } 2$$

> In this question t is the independent variable, and x the dependent variable. The method of solution is the same as in questions connecting x and y.

So the complementary function is $x = Ae^t + Be^{2t}$

The form of the particular integral is $x = \lambda + ut$, so $\frac{dx}{dt} = \mu$ and $\frac{d^2x}{dt^2} = 0$

Substituting into $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2t - 3$ gives:

$-3\mu + 2\lambda + 2\mu t = 2t - 3$

Equate coefficients of t: $2\mu = 2 \Rightarrow \mu = 1$

Equate constant terms: $2\lambda - 3\mu = -3 \Rightarrow \lambda = 0$

So a particular integral is t.

The general solution is $x = Ae^t + Be^{2t} + t$

6 b When $t = 0$, $x = 1$ so $A + B = 1$
When $t = 1$, $x = 2$ so $Ae + Be^2 = 1$

$$Ae + (1 - A)e^2 = 1$$

$$Ae + e^2 - e^2 A = 1$$

$$Ae(1 - e) = 1 - e^2$$

$$A = \frac{1 - e^2}{e(1 - e)}$$

$$A = \frac{1 + e}{e}$$

$$B = 1 - \frac{1 + e}{e}$$

$$B = -\frac{1}{e}$$

So the particular solution is

$$x = \left(\frac{1 + e}{e}\right)e^t + \left(-\frac{1}{e}\right)e^{2t} + t$$

$$= \frac{e^t + e^{t+1}}{e} - \frac{e^{2t}}{e} + t$$

$$= e^{t-1} + e^t - e^{2t-1} + t$$

7 Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} - 9x = 0$

The auxiliary equation is $m^2 - 9 = 0 \Rightarrow m = \pm 3$

So the complementary function is $x = Ae^{3t} + Be^{-3t}$

The form of the particular integral is $x = \lambda \cos t + \mu \sin t$

$\frac{dx}{dt} = -\lambda \sin t + \mu \cos t$ and $\frac{d^2x}{dt^2} = -\lambda \cos t - \mu \sin t$

Substituting into $\frac{d^2x}{dt^2} - 9x = 10\sin t$ gives:

$-\lambda \cos t - \mu \sin t - 9\lambda \cos t - 9\mu \sin t = 10\sin t$

Equating the coefficients of cos t: $-10\lambda = 0 \Rightarrow \lambda = 0$

Equating the coefficients of sin t: $-10\mu = 10 \Rightarrow \mu = -1$

So a particular integral is $-\sin t$

The general solution is $x = Ae^{3t} + Be^{-3t} - \sin t$

The particular integral is of the form $\lambda \cos t + \mu \sin t$.

When $t = 0, x = 2$ so $A + B = 2$ **(1)**

$\frac{dx}{dt} = 3Ae^{3t} - 3Be^{-3t} - \cos t$

When $t = 0, \frac{dx}{dt} = -1$ so $3A - 3B - 1 = -1 \Rightarrow A = B$

Substituting $A = B$ in equation (**1**) gives $2B = 2 \Rightarrow A = B = 1$

Substituting values for A and B into the general solution gives the particular solution

$x = e^{3t} + e^{-3t} - \sin t$

8 a i $x = \lambda t^3 e^{2t}$, so $\frac{dx}{dt} = 3\lambda t^2 e^{2t} + 2\lambda t^3 e^{2t}$

$\frac{d^2x}{dt^2} = 6\lambda t e^{2t} + 6\lambda t^2 e^{2t} + 6\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t} = 6\lambda t e^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t}$

Substituting into $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 3te^{2t}$ gives:

$6\lambda t e^{2t} + 12\lambda t^2 e^{2t} + 4\lambda t^3 e^{2t} - 4(3\lambda t^2 e^{2t} + 2\lambda t^3 e^{2t}) + 4\lambda t^3 e^{2t} = 3te^{2t}$

$\Rightarrow 6\lambda t e^{2t} = 3te^{2t}$

$\Rightarrow \lambda = \frac{1}{2}$

8 **a** **ii** Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2}-4\frac{dx}{dt}+4x=0$

The auxiliary equation is

$$m^2-4m+4=0$$
$$(m-2)(m-2)=0$$
$$m=2$$

So the complementary function is $x=(A+Bt)e^{2t}$

Therefore using the particular integral from part **ai** the general solution is

$$x=(A+Bt)e^{2t}+\frac{1}{2}t^3e^{2t}$$

This equation can also be written as

$$x=Ae^{2t}+Bte^{2t}+\frac{1}{2}t^3e^{2t} \quad \text{or} \quad x=\left(A+Bt+\frac{1}{2}t^3\right)e^{2t}$$

b When $t=0, x=0$ so $A=0$

$$\frac{dx}{dt}=2\left(A+Bt+\frac{1}{2}t^3\right)e^{2t}+\left(B+\frac{3}{2}t^2\right)e^{2t}$$

When $t=0, \frac{dx}{dt}=1$ so $2A+B=1$

As $A=0 \Rightarrow B=1$

Substituting values for A and B into the general solution gives the particular solution

$$x=te^{2t}+\frac{1}{2}t^3e^{2t}=\left(t+\frac{1}{2}t^3\right)e^{2t}=\left(1+\frac{1}{2}t^2\right)te^{2t}$$

9 Solving the corresponding homogeneous equation $25\frac{d^2x}{dt^2}+36x=0$

The auxiliary equation is

$$25m^2+36=0$$
$$m^2=-\frac{36}{25}$$
$$m=\pm\frac{6}{5}i$$

So the complementary function is $x=A\cos\frac{6}{5}t+B\sin\frac{6}{5}t$

The form of the particular integral is $x=\lambda$, so $\frac{dx}{dt}=0$ and $\frac{d^2x}{dt^2}=0$

Substituting into $25\frac{d^2x}{dt^2}+36x=18$ gives:

$$36\lambda=18 \Rightarrow \lambda=\frac{18}{36}=\frac{1}{2}$$

So the general solution is $x=A\cos\frac{6}{5}t+B\sin\frac{6}{5}t+\frac{1}{2}$

9 When $t=0, x=1$ so $A+\frac{1}{2}=1 \Rightarrow A=\frac{1}{2}$

$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{6}{5}A\sin\frac{6}{5}t+\frac{6}{5}B\cos\frac{6}{5}t$

When $t=0, \frac{\mathrm{d}x}{\mathrm{d}t}=0.6$ so $1.2B=0.6 \Rightarrow B=\frac{1}{2}$

Substituting values for A and B into the general solution gives the particular solution

$x=\frac{1}{2}\left(\cos\frac{6}{5}t+\sin\frac{6}{5}t+1\right)$

10 a Solving the corresponding homogeneous equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-2\frac{\mathrm{d}x}{\mathrm{d}t}+2x=0$

The auxiliary equation is

$m^2-2m+2=0$

$m=\frac{2\pm\sqrt{4-8}}{2}=1\pm\mathrm{i}$

So the complementary function is $x=\mathrm{e}^t(A\cos t+B\sin t)$

The form of the particular integral is $x=\upsilon t^2+\mu t+\lambda$, so $\frac{\mathrm{d}x}{\mathrm{d}t}=\mu+2\upsilon t$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=2\upsilon$

Substituting into $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-2\frac{\mathrm{d}x}{\mathrm{d}t}+2x=2t^2$ gives:

$2\upsilon-2(\mu+2\upsilon t)+2(\lambda+\mu t+\upsilon t^2)=2t^2$

Equating the coefficients of t^2: $2\upsilon=2\Rightarrow\upsilon=1$

Equating the coefficients of t: $2\mu-4\upsilon=0\Rightarrow 2\mu=4\upsilon\Rightarrow\mu=2$

Equating constant terms: $2\upsilon-2\mu+2\lambda=0\Rightarrow 2-4+2\lambda=0\Rightarrow 2\lambda=2\Rightarrow\lambda=1$

So a particular integral is t^2+2t+1

The general solution is $x=\mathrm{e}^t(A\cos t+B\sin t)+t^2+2t+1$

b When $t=0, x=1$ so $A+1=1\Rightarrow A=0$

$\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t(A\cos t+B\sin t)+\mathrm{e}^t(-A\sin t+B\cos t)+2t+2$

When $t=0, \frac{\mathrm{d}x}{\mathrm{d}t}=3$ and given $A=0$ so $B+2=3\Rightarrow B=1$

Substituting values for A and B into the general solution gives the particular solution

$x=\mathrm{e}^t\sin t+t^2+2t+1$ which can be rewritten as $x=\mathrm{e}^t\sin t+(1+t)^2$

11 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$
$$m = 1 \text{ or } 2$$

So the complementary function is $y = Ae^x + Be^{2x}$

As the complementary function has an e^{2x} term, try a particular integral in the form $\lambda x e^{2x}$

So $\frac{dy}{dx} = \lambda e^{2x} + 2\lambda x e^{2x}$ and $\frac{d^2y}{dx^2} = 4\lambda e^{2x} + 4\lambda x e^{2x}$

Substituting into $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3e^{2x}$ gives:

$$4\lambda e^{2x} + 4\lambda x e^{2x} - 3(\lambda e^{2x} + 2\lambda x e^{2x}) + 2\lambda x e^{2x} = 3e^{2x}$$
$$\Rightarrow \lambda e^{2x} = 3e^{2x} \Rightarrow \lambda = 3$$

So a particular integral is $3xe^{2x}$

The general solution is $y = Ae^x + Be^{2x} + 3xe^{2x}$

b When $x = 0, y = 0$ so $A + B = 0 \Rightarrow A = -B$

$$\frac{dy}{dx} = Ae^x + 2Be^{2x} + 3e^{2x} + 6xe^{2x}$$

When $x = 0, \frac{dy}{dx} = 0$ and given $A = -B$ so $-B + 2B + 3 = 0 \Rightarrow B = -3$ and $A = 3$

Substituting values for A and B into the general solution from part **a** gives the particular solution

$$y = 3e^x - 3e^{2x} + 3xe^{2x}$$

12 Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} + 9y = 0$

The auxiliary equation is $m^2 + 9 = 0 \Rightarrow m = \pm 3\text{i}$

So the complementary function is $y = A\cos 3x + B\sin 3x$

As the complementary function has a $\sin 3x$ term, try a particular integral $y = \lambda x\cos 3x + \mu x\sin 3x$

$$\frac{dy}{dx} = -3\lambda x\sin 3x + \lambda\cos 3x + 3\mu x\cos 3x + \mu\sin 3x$$

$$\frac{d^2y}{dx^2} = -9\lambda x\cos 3x - 3\lambda\sin 3x - 3\lambda\sin 3x - 9\mu x\sin 3x + 3\mu\cos 3x - 3\mu\cos 3x$$
$$= -9\lambda x\cos 3x - 6\lambda\sin 3x - 9\mu x\sin 3x$$

Substituting into $\frac{d^2y}{dx^2} + 9y = \sin 3x$ gives:

$$-9\lambda x\cos 3x - 6\lambda\sin 3x - 9\mu x\sin 3x + 9\lambda x\cos 3x + 9\mu x\sin 3x = \sin 3x$$
$$\Rightarrow -6\lambda\sin 3x = \sin 3x \Rightarrow \lambda = -\frac{1}{6}$$

12 The equation is satisfied for any μ, so choose $\mu = 0$ and a particular integral is $-\frac{1}{6}x\cos 3x$

The general solution is $y = A\cos 3x + B\sin 3x - \frac{1}{6}x\cos 3x$

When $x = 0, y = 0$ so $A = 0$

$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x - \frac{1}{6}\cos 3x - \frac{1}{2}x\sin 3x$

When $x = 0, \frac{dy}{dx} = 0$ so $3B - \frac{1}{6} = 0 \Rightarrow B = \frac{1}{18}$

Substituting values for A and B into the general solution gives the particular solution

$y = \frac{1}{18}\sin 3x - \frac{1}{6}x\cos 3x$

13 a Solving the corresponding homogeneous equation $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$

The auxiliary equation is

$m^2 + 5m + 6 = 0$

$(m+3)(m+2) = 0$

$m = -3$ or -2

So the complementary function is $x = Ae^{-2t} + Be^{-3t}$

The form of the particular integral is $x = \lambda e^{-t}$, so $\frac{dx}{dt} = -\lambda e^{-t}$ and $\frac{d^2x}{dt^2} = \lambda e^{-t}$

Substituting into $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$ gives:

$\lambda e^{-t} - 5\lambda e^{-t} + 6\lambda e^{-t} = 2e^{-t}$

$\Rightarrow 2\lambda e^{-t} = 2e^{-t} \Rightarrow \lambda = 1$

So a particular integral is e^{-t}

The general solution is $x = Ae^{-2t} + Be^{-3t} + e^{-t}$

When $t = 0, x = 0$ so $A + B + 1 = 0 \Rightarrow A = -B - 1$

$\frac{dx}{dt} = -2Ae^{-2t} - 3Be^{-3t} - e^{-t}$

When $t = 0, \frac{dx}{dt} = 2$ so $-2A - 3B - 1 = 2$

$\Rightarrow -2(-B - 1) - 3B - 1 = 2$

$\Rightarrow B = -1$ and hence $A = 0$

Substituting values for A and B into the general solution gives the particular solution

$x = -e^{-3t} + e^{-t}$

13 b $\frac{\mathrm{d}x}{\mathrm{d}t} = 3\mathrm{e}^{-3t} - \mathrm{e}^{-t}$

When $\frac{\mathrm{d}x}{\mathrm{d}t} = 0, 3\mathrm{e}^{-3t} = \mathrm{e}^{-t}$ and taking logarithms of both sides gives $\ln 3\mathrm{e}^{-3t} = \ln^{-t}$

$\Rightarrow \ln 3 + \ln \mathrm{e}^{-3t} = \ln \mathrm{e}^{-t} \Rightarrow \ln 3 - 3t = -t$

$\Rightarrow t = \frac{1}{2}\ln 3$

$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9\mathrm{e}^{-3t} + \mathrm{e}^{-t}$, so when $t = \frac{1}{2}\ln 3$

$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9\mathrm{e}^{-\frac{3}{2}\ln 3} + \mathrm{e}^{-\frac{1}{2}\ln 3} = -9\mathrm{e}^{-\ln 3^{-\frac{3}{2}}} + \mathrm{e}^{\ln 3^{-\frac{1}{2}}} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

So when $t = \frac{1}{2}\ln 3$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} < 0$, there is a maximum, and as the function is continuous this is its maximum value. Substituting in the equation gives

Maximum value $= -\mathrm{e}^{-\frac{3}{2}\ln 3} + \mathrm{e}^{-\frac{1}{2}\ln 3} = -\mathrm{e}^{\ln 3^{-\frac{3}{2}}} + \mathrm{e}^{\ln 3^{-\frac{1}{2}}} = -3^{-\frac{3}{2}} + 3^{-\frac{1}{2}} = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$