Methods in differential equations 7C

1 a First consider the corresponding homogeneous equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=0$

The auxiliary equation is

$m^2+6m+5=0$

$(m+5)(m+1)=0$

$m=-5 \text{ or } -1$

So the complementary function is $y=Ae^{-x}+Be^{-5x}$

The form of the particular integral is $y=\lambda$, so $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=0$

Substituting into $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=10$ gives:

$5\lambda=10 \Rightarrow \lambda=2$

So the general solution is $y=Ae^{-x}+Be^{-5x}+2$

b First consider the corresponding homogeneous equation $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+12y=0$

The auxiliary equation is

$m^2-8m+12=0$

$(m-6)(m-2)=0$

$m=6 \text{ or } 2$

So the complementary function is $y=Ae^{6x}+Be^{2x}$

The form of the particular integral is $y=\lambda+\mu x$, so $\frac{dy}{dx}=\mu$, $\frac{d^2y}{dx^2}=0$

Substituting into $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+12y=36x$ gives:

$-8\mu+12\lambda+12\mu x=36x$

Comparing coefficients of x: $12\mu=36 \Rightarrow \mu=3$

Comparing constants: $-8\mu+12\lambda=0 \Rightarrow 3\lambda=2\mu$

Substituting for μ: $3\lambda=6 \Rightarrow \lambda=2$

So a particular integral is $2+3x$

The general solution is $y=Ae^{6x}+Be^{2x}+2+3x$

1 c Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=0$

The auxiliary equation is

$$m^2+m-12=0$$
$$(m+4)(m-3)=0$$
$$m=-4 \text{ or } 3$$

So the complementary function is $y=Ae^{-4x}+Be^{3x}$

The form of the particular integral is $y=\lambda e^{2x}$, so $\frac{dy}{dx}=2\lambda e^{2x}$ and $\frac{d^2y}{dx^2}=4\lambda e^{2x}$

Substituting into $\frac{d^2y}{dx^2}+\frac{dy}{dx}-12y=12e^{2x}$ gives:

$$4\lambda e^{2x}+2\lambda e^{2x}-12\lambda e^{2x}=12e^{2x}$$
$$\Rightarrow -6\lambda e^{2x}=12e^{2x}\Rightarrow \lambda=-2$$

So a particular integral is $-2e^{2x}$

The general solution is $y=Ae^{-4x}+Be^{3x}-2e^{2x}$

d Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=0$

The auxiliary equation is

$$m^2+2m-15=0$$
$$(m+5)(m-3)=0$$
$$m=-5 \text{ or } 3$$

So the complementary function is $y=Ae^{-5x}+Be^{3x}$

The form of the particular integral is $y=\lambda$, so $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=0$

Substituting into $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-15y=5$ gives:

$$-15\lambda=5\Rightarrow \lambda=-\frac{1}{3}$$

So a particular integral is $-\frac{1}{3}$

The general solution is $y=Ae^{-5x}+Be^{3x}-\frac{1}{3}$

1 e Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

The auxiliary equation is

$$m^2 - 8m + 16 = 0$$
$$(m-4)(m-4) = 0$$
$$m = 4$$

The auxiliary equation has a repeated root so the complementary function is of the form $(A + Bx)e^{\alpha x}$

So the complementary function is $y = (A + Bx)e^{4x}$

The particular integral is $y = \lambda + \mu x$, so $\frac{dy}{dx} = \mu$ and $\frac{d^2y}{dx^2} = 0$

Substituting in $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 8x + 12$ gives:

$-8\mu + 16\lambda + 16\mu x = 8x + 12$

Comparing coefficients of x: $16\mu = 8 \Rightarrow \mu = \frac{1}{2}$

Comparing constants: $-8\mu + 16\lambda = 12 \Rightarrow 4\lambda = 3 + 2\mu$

Substituting for μ: $4\lambda = 3 + 1 \Rightarrow \lambda = 1$

So a particular integral is $1 + \frac{1}{2}x$

The general solution is $y = (A + Bx)e^{4x} + 1 + \frac{1}{2}x$

1 f Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

The auxiliary equation is

$$m^2+2m+1=0$$
$$(m+1)(m+1)=0$$
$$m=-1$$

So the complementary function is $y=(A+Bx)e^{-x}$

The particular integral is $y=\lambda\cos 2x+\mu\sin 2x$, so

$$\frac{dy}{dx}=-2\lambda\sin 2x+2\mu\cos 2x$$
$$\frac{d^2y}{dx^2}=-4\lambda\cos 2x-4\mu\sin 2x$$

Substituting in $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=25\cos 2x$ gives:

$$-4\lambda\cos 2x-4\mu\sin 2x+2(-2\lambda\sin 2x+2\mu\cos 2x)+\lambda\cos 2x+\mu\sin 2x=25\cos 2x$$
$$\Rightarrow(-4\lambda+4\mu+\lambda)\cos 2x+(-4\mu-4\lambda+\mu)\sin 2x=25\cos 2x$$
$$\Rightarrow(4\mu-3\lambda)\cos 2x-(3\mu+4\lambda)\sin 2x=25\cos 2x$$

Equating the coefficients of cos 2x: $4\mu-3\lambda=25$ **(1)**

Equating the coefficients of sin 2x: $3\mu+4\lambda=0$ **(2)**

Adding 4 × equation (**1**) to 3 × equation (**2**) gives:

$16\mu-12\lambda+9\mu+12\lambda=100\Rightarrow 25\mu=100\Rightarrow\mu=4$

Substituting for in equation (**2**) gives: $3\times4+4\lambda=0\Rightarrow\lambda=-3$

So a particular integral is $y=4\sin 2x-3\cos 2x$

The general solution is $y=(A+Bx)e^{-x}+4\sin 2x-3\cos 2x$.

g Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+81y=0$

> If the auxiliary equation has imaginary roots, the complementary function is of the form $A\cos\omega x+B\sin\omega x$

The auxiliary equation is

$$m^2+81=0$$
$$m=\pm 9i$$

The complementary function is $y=A\cos 9x+B\sin 9x$

The particular integral is $y=\lambda e^{3x}$, so $\frac{dy}{dx}=3\lambda e^{3x}$ and $\frac{d^2y}{dx^2}=9\lambda e^{3x}$

Substituting into $\frac{d^2y}{dx^2}+81y=15e^{3x}$ gives:

$$9\lambda e^{3x}+81\lambda e^{3x}=15e^{3x}\Rightarrow 90\lambda e^{3x}=15e^{3x}\Rightarrow\lambda=\frac{15}{90}=\frac{1}{6}$$

So a particular integral is $\frac{1}{6}e^{3x}$

The general solution is $y=A\cos 9x+B\sin 9x+\frac{1}{6}e^{3x}$

1 h Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}+4y=0$

The auxiliary equation is

$m^2+4=0$

$m=\pm 2i$

The complementary function is $y=A\cos 2x+B\sin 2x$

The particular integral is $y=\lambda\cos x+\mu\sin x$

$\frac{dy}{dx}=-\lambda\sin x+\mu\cos x$

$\frac{d^2y}{dx^2}=-\lambda\cos x-\mu\sin x$

Substituting into $\frac{d^2y}{dx^2}+4y=\sin x$ gives:

Then $-\lambda\cos x-\mu\sin x+4(\lambda\cos x+\mu\sin x)=\sin x$

Equating the coefficients of cos x: $3\lambda=0\Rightarrow\lambda=0$

Equating the coefficients of sin x: $3\mu=1\Rightarrow\mu=\frac{1}{3}$

So a particular integral is $\frac{1}{3}\sin x$

The general solution is $y=A\cos 2x+B\sin 2x+\frac{1}{3}\sin x$

i Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=0$

The auxiliary equation is

$m^2-4m+5=0$

$m=\frac{4\pm\sqrt{16-20}}{2}=2\pm i$

If the auxiliary equation has complex roots, the complementary function is of the form $e^{px}(A\cos ax+B\sin ax)$

The complementary function is $y=e^{2x}(A\cos x+B\sin x)$

The particular integral is $y=\lambda+\mu x+\upsilon x^2$, so $\frac{dy}{dx}=\mu+2\upsilon x$ and $\frac{d^2y}{dx^2}=2\upsilon$

Substituting into $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+5y=25x^2-7$ gives:

$2\upsilon-4(\mu+2\upsilon x)+5(\lambda+\mu x+\upsilon x^2)=25x^2-7$

$\Rightarrow 5\upsilon x^2+(5\mu-8\upsilon)x+2\upsilon-4\mu+5\lambda=25x^2-7$

Equating the coefficients of x^2: $5\upsilon=25\Rightarrow\upsilon=5$

Equating the coefficients of x: $5\mu-8\upsilon=0\Rightarrow 5\mu=8\upsilon=40\Rightarrow\mu=8$

Equating constant terms: $2\upsilon-4\mu+5\lambda=-7\Rightarrow 10-32+5\lambda=-7\Rightarrow 5\lambda=15\Rightarrow\lambda=3$

So the particular integral is $3+8x+5x^2$

The general solution is $y=e^{2x}(A\cos x+B\sin x)+3+8x+5x^2$

1 j Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+26y=0$

The auxiliary equation is

$m^2-2m+26=0$

$m=\frac{2\pm\sqrt{4-4\times 26}}{2}=\frac{2\pm\sqrt{-100}}{2}=1\pm 5i$

The complementary function is $y=e^x(A\cos 5x+B\sin 5x)$

The particular integral is $y=\lambda e^x$, so $\frac{dy}{dx}=\lambda e^x$ and $\frac{d^2y}{dx^2}=\lambda e^x$

Substitute into $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+26y=e^x$

$\lambda e^x-2\lambda e^x+26\lambda e^x=1$

$\Rightarrow 25\lambda=1\Rightarrow\lambda=\frac{1}{25}$

So the particular integral is $\frac{1}{25}e^x$

The general solution is $y=e^x(A\cos 5x+B\sin 5x)+\frac{1}{25}e^x$

2 a Consider a particular integral of the form $y=\upsilon x^2+\mu x+\lambda$, so $\frac{dy}{dx}=\mu+2\upsilon x$ and $\frac{d^2y}{dx^2}=2\upsilon$

Substituting into $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=x^2-3x+2$ gives:

$2\upsilon-5(\mu+2\upsilon x)+4(\lambda+\mu x+\upsilon x^2)=x^2-3x+2$

$\Rightarrow 4\upsilon x^2+(4\mu-10\upsilon)x+2\upsilon-5\mu+4\lambda=x^2-3x+2$

Equating the coefficients of x^2: $4\upsilon=1\Rightarrow\upsilon=\frac{1}{4}$

Equating the coefficients of x: $4\mu-10\upsilon=-3\Rightarrow 4\mu=10\upsilon-3\Rightarrow\mu=-\frac{1}{8}$

Equating constant terms: $2\upsilon-5\mu+4\lambda=2\Rightarrow\frac{1}{2}+\frac{5}{8}+4\lambda=2\Rightarrow 4\lambda=\frac{7}{8}\Rightarrow\lambda=\frac{7}{32}$

So the particular integral is $\frac{1}{4}x^2-\frac{1}{8}x+\frac{7}{32}$

b Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+26y=0$

The auxiliary equation is

$m^2-5m+4=0$

$(m-4)(m-1)=0$

$m=1$ or 4

So the complementary function is $y=Ae^{4x}+Be^x$

The general solution of the given equation is $y=Ae^{4x}+Be^x+\frac{1}{4}x^2-\frac{1}{8}x+\frac{7}{32}$

3 a The complementary function is the general solution of the equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0$

The auxiliary equation is

$m^2 - 6m = 0$

$m(m-6) = 0$

$m = 0 \text{ or } 6$

So the complementary function is $y = Ae^{6x} + Be^{0x} = Ae^{6x} + B$

b As the complementary function includes a constant term, multiply the 'expected' particular integral by x, so consider a particular integral of the form $y = \upsilon x^3 + \mu x^2 + \lambda x$

$\frac{dy}{dx} = 3\upsilon x^2 + 2\mu x + \lambda$ and $\frac{d^2y}{dx^2} = 6\upsilon x + 2\mu$

Substituting into $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$ gives:

$6\upsilon x + 2\mu - 6(3\upsilon x^2 + 2\mu x + \lambda) = 2x^2 - x + 1$

$\Rightarrow -18\upsilon x^2 + (6\upsilon - 12\mu)x + 2\mu - 6\lambda = 2x^2 - x + 1$

Equating the coefficients of x^2: $-18\upsilon = 2 \Rightarrow \upsilon = -\frac{1}{9}$

Equating the coefficients of x: $6\upsilon - 12\mu = -1 \Rightarrow 12\mu = 6\upsilon + 1 \Rightarrow 12\mu = \frac{1}{3} \Rightarrow \mu = \frac{1}{36}$

Equating constant terms: $2\mu - 6\lambda = 1 \Rightarrow 6\lambda = -\frac{34}{36} \Rightarrow \lambda = \frac{17}{108}$

So the particular integral is $-\frac{1}{9}x^3 + \frac{1}{36}x^2 + \frac{17}{108}x$

The general solution is $y = Ae^{6x} + B - \frac{1}{9}x^3 + \frac{1}{36}x^2 + \frac{17}{108}x$

4 The complementary function is the general solution of the equation $\frac{d^2y}{dx^2}+4\frac{dy}{dx}=0$

The auxiliary equation is

$$m^2+4m=0$$
$$m(m+4)=0$$
$$m=0 \text{ or } -4$$

So the complementary function is $y=A+Be^{-4x}$

As the complementary function includes a constant term, multiply the 'expected' particular integral by x, so consider a particular integral of the form $y=\upsilon x^3+\mu x^2+\lambda x$

$\frac{dy}{dx}=3\upsilon x^2+2\mu x+\lambda$ and $\frac{d^2y}{dx^2}=6\upsilon x+2\mu$

Substituting into $\frac{d^2y}{dx^2}+4\frac{dy}{dx}=24x^2$ gives:

$$6\upsilon x+2\mu+4(3\upsilon x^2+2\mu x+\lambda)=24x^2$$
$$\Rightarrow 12\upsilon x^2+(6\upsilon+8\mu)x+2\mu+4\lambda=24x^2$$

Equating the coefficients of x^2: $12\upsilon=24\Rightarrow\upsilon=2$

Equating the coefficients of x: $6\upsilon+8\mu=0\Rightarrow 8\mu=-12\Rightarrow\mu=-\frac{3}{2}$

Equating constant terms: $2\mu+4\lambda=0\Rightarrow 4\lambda=3\Rightarrow\lambda=\frac{3}{4}$

So the particular integral is $2x^3-\frac{3}{2}x^2+\frac{3}{4}x$

The general solution is $y=A+Be^{-4x}+2x^3-\frac{3}{2}x^2+\frac{3}{4}x$

5 a Solving the corresponding homogeneous equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$

The auxiliary equation is

$$m^2-2m+1=0$$
$$(m-1)(m-1)=0$$
$$m=1$$

So the complementary function is $y=(A+Bx)e^x$

The complementary function contains an xe^x and so λxe^x is not a suitable form for the particular integral of this equation.

Note that if $y=\lambda xe^x$, then $\frac{dy}{dx}=\lambda e^x+\lambda xe^x$ and $\frac{d^2y}{dx^2}=2\lambda e^x+\lambda xe^x$

Substituting into $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x$ gives:

$$2\lambda e^x+\lambda xe^x-2(\lambda e^x+\lambda xe^x)+\lambda xe^x=e^x$$

$\Rightarrow 0=e^x$, which is impossible as $e^x>0$

5 b if $y = \lambda x^2 e^x$, then $\dfrac{dy}{dx} = 2\lambda x e^x + \lambda x^2 e^x$ and $\dfrac{d^2y}{dx^2} = 2\lambda e^x + 4\lambda x e^x + \lambda x^2 e^x$

Substituting into $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x$ gives:

$2\lambda e^x + 4\lambda x e^x + \lambda x^2 e^x - 2(2\lambda x e^x + \lambda x^2 e^x) + \lambda x^2 e^x = e^x$

$\Rightarrow 2\lambda e^x = e^x \Rightarrow \lambda = \dfrac{1}{2}$

c The general solution is the complementary function (from part **a**) plus the particular integral (from part **b**), so it is

$$y = (A + Bx)e^x + \frac{1}{2}x^2 e^x = \left(A + Bx + \frac{1}{2}x^2\right)e^x$$

6 a Solving the corresponding homogeneous equation $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = 0$

The auxiliary equation is

$m^2 + 4m + 3 = 0$

$(m+3)(m+1) = 0$

$m = -1 \text{ or } -3$

So the complementary function is $y = Ae^{-t} + Be^{-3t}$

The form of the particular integral is $y = \lambda + \mu t$, so $\dfrac{dy}{dt} = \mu$, $\dfrac{d^2y}{dt^2} = 0$

Substituting into $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = kt + 5$ gives:

$4\mu + 3\lambda + 3\mu t = kt + 5$

Equating the coefficients of t: $3\mu = k \Rightarrow \mu = \dfrac{k}{3}$

Equating constant terms: $4\mu + 3\lambda = 5 \Rightarrow 3\lambda = 5 - \dfrac{4k}{3} \Rightarrow \lambda = \dfrac{5}{3} - \dfrac{4k}{9}$

So the particular integral is $\dfrac{5}{3} - \dfrac{4k}{9} + \dfrac{k}{3}t$

The general solution is $y = Ae^{-t} + Be^{-3t} + \dfrac{5}{3} - \dfrac{4k}{9} + \dfrac{kt}{3}$

b If $k = 6$, then the general solution is $y = Ae^{-t} + Be^{-3t} + 2t - 1$.

As $t \to \infty$, $e^{-t} \to 0$, so for large values of t the general solution may be approximated by $y = 2t - 1$

Challenge

Solving the corresponding homogeneous equation $\frac{d^2y}{dt^2} + y = 0$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

So the complementary function is $y = A\cos x + B\sin x$

To find a particular integral, consider functions of the form $y = \lambda x e^{2x} + \mu e^{2x}$

$\frac{dy}{dx} = \lambda e^{2x} + 2\lambda x e^{2x} + 2\mu e^{2x}$

$\frac{d^2y}{dx^2} = 2\lambda e^{2x} + 2\lambda e^{x} + 4\lambda x e^{2x} + 4\mu e^{2x} = 4\lambda e^{2x} + 4\lambda x e^{2x} + 4\mu e^{2x}$

Substituting into $\frac{d^2y}{dx^2} + y = 5xe^{2x}$ gives:

$4\lambda e^{2x} + 4\lambda x e^{2x} + 4\mu e^{2x} + \lambda x e^{2x} + \mu e^{2x} = 5xe^{2x}$

$\Rightarrow (4\lambda + 5\mu)e^{2x} + 5\lambda x e^{2x} = 5xe^{2x}$

Equating the coefficients of xe^{2x}: $5\lambda = 5 \Rightarrow \lambda = 1$

Equating the coefficients of e^{2x}: $4\lambda + 5\mu = 0 \Rightarrow \mu = -\frac{4}{5}$

So the particular integral is $xe^{2x} - \frac{4}{5}e^{2x}$

The general solution is $y = A\cos x + B\sin x + xe^{2x} - \frac{4}{5}e^{2x}$