Methods in differential equations 7B

1 a The auxiliary equation is

$m^2+5m+6=0$

$(m+3)(m+2)=0$

$m=-3$ or -2

So the general solution is $y=Ae^{-3x}+Be^{-2x}$

The auxiliary equation of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ is $am^2+bm^2+c=0$. If α and β are real roots of this quadratic then $y=Ae^{\alpha x}+Be^{\beta x}$ is the general solution of the differential equation.

b The auxiliary equation is

$m^2-8m+12=0$

$(m-6)(m-2)=0$

$m=2$ or 6

So the general solution is $y=Ae^{2x}+Be^{6x}$

c The auxiliary equation is

$m^2+2m-15=0$

$(m+5)(m-3)=0$

$m=-5$ or 3

So the general solution is $y=Ae^{-5x}+Be^{3x}$

d The auxiliary equation is

$m^2-3m-28=0$

$(m-7)(m+4)=0$

$m=7$ or -4

So the general solution is $y=Ae^{7x}+Be^{-4x}$

e The auxiliary equation is

$m^2+5m=0$

$m(m+5)=0$

$m=0$ or -5

The auxiliary equation has two real roots, but one of them is zero. As $e^{0x}=1$, the general solution is $y=Ae^{0x}+Be^{-5x}=A+Be^{-5x}$

f The auxiliary equation is

$3m^2+7m+2=0$

$(3m+1)(m+2)=0$

$m=-\frac{1}{3}$ or -2

So the general solution is $y=Ae^{-\frac{1}{3}x}+Be^{-2x}$

1 g The auxiliary equation is

$$4m^2 - 7m - 2 = 0$$
$$(4m+1)(m-2) = 0$$
$$m = -\frac{1}{4} \text{ or } 2$$

So the general solution is $y = Ae^{-\frac{1}{4}x} + Be^{2x}$

h The auxiliary equation is

$$15m^2 - 7m - 2 = 0$$
$$(5m+1)(3m-2) = 0$$
$$m = -\frac{1}{5} \text{ or } \frac{2}{3}$$

So the general solution is $y = Ae^{-\frac{1}{5}x} + Be^{\frac{2}{3}x}$

2 a The auxiliary equation is

$$m^2 + 10m + 25 = 0$$
$$(m+5)(m+5) = 0$$
$$m = -5$$

So the general solution is $y = (A + Bx)e^{-5x}$

> The auxiliary equation of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $am^2 + bm^2 + c = 0$. If this equation has one repeated real root α then the general solution of the differential equation is $y = (A + Bx)e^{\alpha x}$

b The auxiliary equation is

$$m^2 - 18m + 81 = 0$$
$$(m-9)(m-9) = 0$$
$$m = 9$$

So the general solution is $y = (A + Bx)e^{9x}$

c The auxiliary equation is

$$m^2 + 2m + 1 = 0$$
$$(m+1)(m+1) = 0$$
$$m = -1$$

So the general solution is $y = (A + Bx)e^{-x}$

d The auxiliary equation is

$$m^2 - 8m + 16 = 0$$
$$(m-4)(m-4) = 0$$
$$m = 4$$

So the general solution is $y = (A + Bx)e^{4x}$

2 e The auxiliary equation is

$$16m^2 + 8m + 1 = 0$$
$$(4m+1)(4m+1) = 0$$
$$m = -\frac{1}{4}$$

So the general solution is $y = (A + Bx)e^{-\frac{1}{4}x}$

f The auxiliary equation is

$$4m^2 - 4m + 1 = 0$$
$$(2m-1)(2m-1) = 0$$
$$m = \frac{1}{2}$$

So the general solution is $y = (A + Bx)e^{\frac{1}{2}x}$

g The auxiliary equation is

$$4m^2 + 20m + 25 = 0$$
$$(2m+5)(2m+5) = 0$$
$$m = -\frac{5}{2}$$

So the general solution is $y = (A + Bx)e^{-\frac{5}{2}x}$

h The auxiliary equation is

$$m^2 + 2\sqrt{3}m + 3 = 0$$
$$(m+\sqrt{3})(m+\sqrt{3}) = 0$$
$$m = -\sqrt{3}$$

So the general solution is $y = (A + Bx)e^{-\sqrt{3}x}$

3 a The auxiliary equation is

$$m^2 + 25 = 0$$
$$\Rightarrow m = \pm 5\text{i}$$

The general solution is $y = A\cos 5x + B\sin 5x$

If the auxiliary equation has purely imaginary roots, the general solution has the form $y = A\cos\omega x + B\sin\omega x$, where A and B are constants and $\text{i}\omega$ is the solution of the auxiliary equation.

b The auxiliary equation is

$$m^2 + 81 = 0$$
$$\Rightarrow m = \pm 9\text{i}$$

The general solution is $y = A\cos 9x + B\sin 9x.$

3 c The auxiliary equation is

$m^2 + 1 = 0$

$\Rightarrow m = \pm \mathrm{i}$

The general solution is $y = A\cos x + B\sin x$

d The auxiliary equation is

$9m^2 + 16 = 0$

$m^2 = -\frac{16}{9}$

$\Rightarrow m = \pm\frac{4}{3}\mathrm{i}$

The general solution is $y = A\cos\frac{4}{3}x + B\sin\frac{4}{3}x$

e The auxiliary equation is

$m^2 + 8m + 17 = 0$

$m = \frac{-8 \pm \sqrt{64 - 4\times 17}}{2} = -4 \pm \frac{1}{2}\sqrt{-4} = -4 \pm \mathrm{i}$ using the quadratic formula

The general solution is $y = \mathrm{e}^{-4x}(A\cos x + B\sin x)$

If the auxiliary equation has complex roots, the general solution has the form $y = \mathrm{e}^{px}(A\cos qx + B\sin qx)$, where A and B are constants and $p \pm \mathrm{i}q$ are solutions of the auxiliary equation.

f The auxiliary equation is

$m^2 - 4m + 5 = 0$

$m = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm \frac{1}{2}\sqrt{-4} = 2 \pm \mathrm{i}$

The general solution is $y = \mathrm{e}^{2x}(A\cos x + B\sin x)$

g The auxiliary equation is

$m^2 + 20m + 109 = 0$

$m = \frac{-20 \pm \sqrt{400 - 436}}{2} = \frac{-20 \pm \sqrt{-36}}{2} = -10 \pm 3\mathrm{i}$

The general solution is $y = \mathrm{e}^{-10x}(A\cos 3x + B\sin 3x)$

h The auxiliary equation is

$m^2 + \sqrt{3}m + 3 = 0$

$m = \frac{-\sqrt{3} \pm \sqrt{3 - 12}}{2} = \frac{-\sqrt{3} \pm \sqrt{-9}}{2} = \frac{-\sqrt{3} \pm 3\mathrm{i}}{2}$

The general solution is $y = \mathrm{e}^{-\frac{\sqrt{3}}{2}x}\left(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x\right)$

4 a The auxiliary equation is

$$m^2 + 14m + 49 = 0$$
$$(m+7)(m+7) = 0$$
$$m = -7$$

So the general solution is $y = (A + Bx)e^{-7x}$

b The auxiliary equation is

$$m^2 + m - 12 = 0$$
$$(m+4)(m-3) = 0$$
$$m = -4 \text{ or } 3$$

So the general solution is $y = Ae^{-4x} + Be^{3x}$

c The auxiliary equation is

$$m^2 + 4m + 13 = 0$$
$$m = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3\text{i}$$

The general solution is $y = e^{-2x}(A\cos 3x + B\sin 3x)$

d The auxiliary equation is

$$16m^2 - 24m + 9 = 0$$
$$(4m-3)(4m-3) = 0$$
$$m = \frac{3}{4}$$

So the general solution is $y = (A + Bx)e^{\frac{3}{4}x}$

e The auxiliary equation is

$$9m^2 - 6m + 5 = 0$$
$$m = \frac{6 \pm \sqrt{36 - 4\times 9\times 5}}{2\times 9} = \frac{6 \pm \sqrt{36-180}}{18} = \frac{6 \pm \sqrt{-144}}{18} = \frac{6 \pm 12\text{i}}{18} = \frac{1 \pm 2\text{i}}{3}$$

So the general solution is $y = e^{\frac{1}{3}x}\left(A\cos\frac{2}{3}x + B\sin\frac{2}{3}x\right)$

f The auxiliary equation is

$$6m^2 - m - 2 = 0$$
$$(3m-2)(2m+1) = 0$$
$$m = \frac{2}{3} \text{ or } -\frac{1}{2}$$

So the general solution is $y = Ae^{\frac{2}{3}x} + Be^{-\frac{1}{2}x}$

5 a The auxiliary equation is

$m^2 + 2km + 9 = 0$

$\Rightarrow m = \dfrac{-2k \pm \sqrt{4k^2 - 36}}{2} = -k \pm \sqrt{k^2 - 9}$ using the quadratic formula

i If $|k| > 3$, the auxiliary equation has two real solutions

So the differential equation has the general solution $x = A\mathrm{e}^{(-k+\sqrt{k^2-9})t} + B\mathrm{e}^{(-k-\sqrt{k^2-9})t}$

ii If $|k| < 3$, then the auxiliary equation has two complex conjugate roots, $-k \pm \mathrm{i}\sqrt{9-k^2}$

So the general solution is $x = \mathrm{e}^{-kt}\left(A\cos\left(\left(\sqrt{9-k^2}\right)t\right) + B\sin\left(\left(\sqrt{9-k^2}\right)t\right)\right)$

iii If $|k| = 3$, then the solution of the auxiliary equation is one repeated root –k,

So the differential equation has general solution $x = (A + Bt)\mathrm{e}^{-kt}$;

b i The general solution can be found simply substituting k = 2 in the equation found in part **a ii**

So the general solution is $x = \mathrm{e}^{-2t}\left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)$

ii When $t \to \infty$, $\mathrm{e}^{-2t} \to 0$, while the trigonometric functions take values that are bounded, so $x \to 0$.

6 If α is the only root of the equation, then:

(1) $a\alpha^2 + b\alpha + c = 0$ by definition

(2) $\alpha = -\dfrac{b}{2a}$ from the quadratic formula, as equal roots $\Rightarrow b^2 = 4ac$

Let $y = (A + Bx)\mathrm{e}^{\alpha x}$

Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = B\mathrm{e}^{\alpha x} + (A + Bx)\alpha\mathrm{e}^{\alpha x}$

And $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = B\alpha\mathrm{e}^{\alpha x} + B\alpha\mathrm{e}^{\alpha x} + (A + Bx)\alpha^2\mathrm{e}^{\alpha x} = 2B\alpha\mathrm{e}^{\alpha x} + (A + Bx)\alpha^2\mathrm{e}^{\alpha x}$

Substituting these results into the differential equation gives:

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + c = a(2B\alpha\mathrm{e}^{\alpha x} + (A + Bx)\alpha^2\mathrm{e}^{\alpha x}) + b(B\mathrm{e}^{\alpha x} + (A + Bx)\alpha\mathrm{e}^{\alpha x}) + c(A + Bx)\mathrm{e}^{\alpha x}$$
$$= B\mathrm{e}^{\alpha x}(2a\alpha + b) + (A + Bx)\mathrm{e}^{\alpha x}(a\alpha^2 + b\alpha + c)$$

But from equation **(1)** $a\alpha^2 + b\alpha + c = 0$ and from equation **(2)** $\alpha = -\dfrac{b}{2a} \Rightarrow 2a\alpha + b = 0$

Hence $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + c = 0$, and so $y = (A + Bx)\mathrm{e}^{\alpha x}$ is a solution to this equation.

7 Let $y = A\mathrm{f}(x) + B\mathrm{g}(x)$

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = A\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + B\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x}$

And $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = A\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + B\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2}$

Then substituting these results into the differential equation gives:

$$a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + c = a\left(A\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + B\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2}\right) + b\left(A\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + B\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x}\right) + c\big(A\mathrm{f}(x) + B\mathrm{g}(x)\big)$$

$$= A\left(a\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + c\,\mathrm{f}(x)\right) + B\left(a\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x} + c\,\mathrm{g}(x)\right)$$

As $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are solutions of the differential equation, it follows that

$$a\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + c\,\mathrm{f}(x) = 0 \quad \text{and} \quad a\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x} + c\,\mathrm{g}(x) = 0$$

Therefore $A\left(a\frac{\mathrm{d}^2\,\mathrm{f}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{f}(x)}{\mathrm{d}x} + c\,\mathrm{f}(x)\right) + B\left(a\frac{\mathrm{d}^2\,\mathrm{g}(x)}{\mathrm{d}x^2} + b\frac{\mathrm{d}\,\mathrm{g}(x)}{\mathrm{d}x} + c\,\mathrm{g}(x)\right) = 0$

So $y = A\mathrm{f}(x) + B\mathrm{g}(x)$ is a solution.

Challenge

If a real-valued quadratic equation has complex roots $p \pm q\mathrm{i}$ then by Euler's formula,

$$\mathrm{e}^{p+\mathrm{i}q} = \mathrm{e}^p\mathrm{e}^{\mathrm{i}q} = \mathrm{e}^p(\cos q + \mathrm{i}\sin q)$$
$$\mathrm{e}^{p-\mathrm{i}q} = \mathrm{e}^p\mathrm{e}^{\mathrm{i}q} = \mathrm{e}^p(\cos(-q) + \mathrm{i}\sin(-q)) = \mathrm{e}^p(\cos q - \mathrm{i}\sin q)$$

So substituting for $\alpha = p + \mathrm{i}q$ and $\beta = p - \mathrm{i}q$ into $Ae^{\alpha x} + Be^{\beta x}$ gives:

$$Ae^{\alpha x} + Be^{\beta x} = A\mathrm{e}^{px}(\cos qx + \mathrm{i}\sin qx) + B\mathrm{e}^{px}(\cos qx - \mathrm{i}\sin qx)$$
$$= \mathrm{e}^{px}((A + B)\cos qx + (A - B)\mathrm{i}\sin qx)$$

Choose A and B such that they are complex conjugates, i.e. there are real numbers λ and μ where $A = \lambda + \mu\mathrm{i}$ and $B = \lambda - \mu\mathrm{i}$

So $A + B = 2\lambda$ and $(A - B)\mathrm{i} = (2u\mathrm{i})\mathrm{i} = 2u\mathrm{i}^2 = -2u$

Hence $Ae^{\alpha x} + Be^{\beta x} = \mathrm{e}^{px}(2\lambda\cos qx - 2\mu\sin qx) = \mathrm{e}^{px}(C\cos qx + D\sin qx)$,
where C and D are real constants, $C = 2\lambda$ and $D = -2\mu$