Methods in differential equations 7A

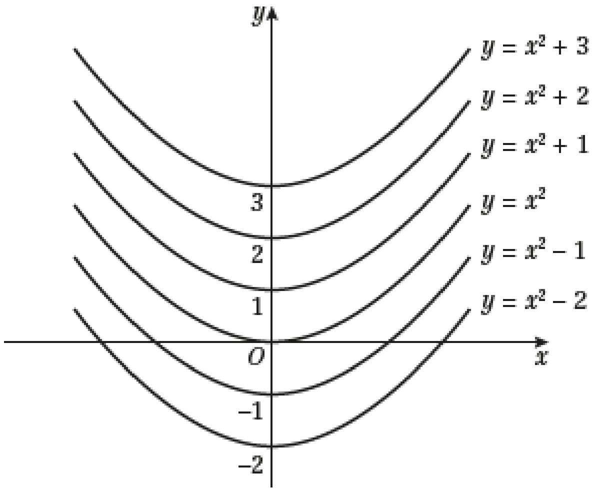
1 a Integrating both sides of the equation and including a constant:

$$\frac{dy}{dx} = 2x$$

$$\Rightarrow y = \int 2x\,dx$$

$$\Rightarrow y = x^2 + c \quad \text{where } c \text{ is a constant}$$

The family of solution curves are parabola.
Sketching the solution curves for $c = -2, -1, 0, 1, 2$ and 3 gives:



1 b Separating the variables and integrating:

$$\frac{dy}{dx} = y$$

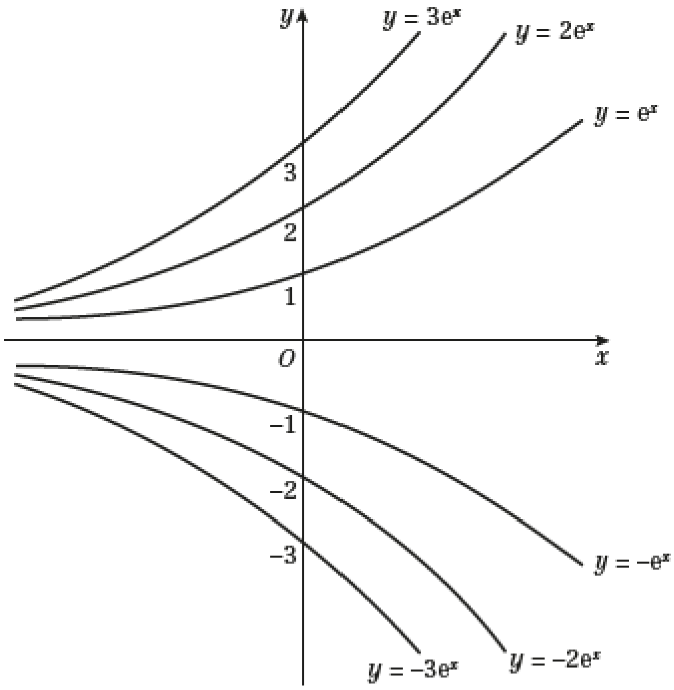
$$\Rightarrow \int \frac{1}{y} dy = \int 1 dx$$

$$\Rightarrow \ln y = x + c \quad \text{where } c \text{ is a constant}$$

$$\Rightarrow y = e^{x+c} = e^c \times e^x$$

$$\Rightarrow y = Ae^x \quad \text{where } A \text{ is a constant } (A = e^c)$$

The family of solution curves are exponential curves.
Sketching the solution curves for $A = -3, -2, -1, 1, 2$ and 3 gives:



1 c Separating the variables and integrating:

$\frac{dy}{dx} = \frac{2y}{x}$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{2}{x} dx$

$\Rightarrow \ln y = 2\ln x + c$

Expressing the constant as ln*A* and simplifying using the laws of logarithms:

$\ln y = 2\ln x + \ln A$

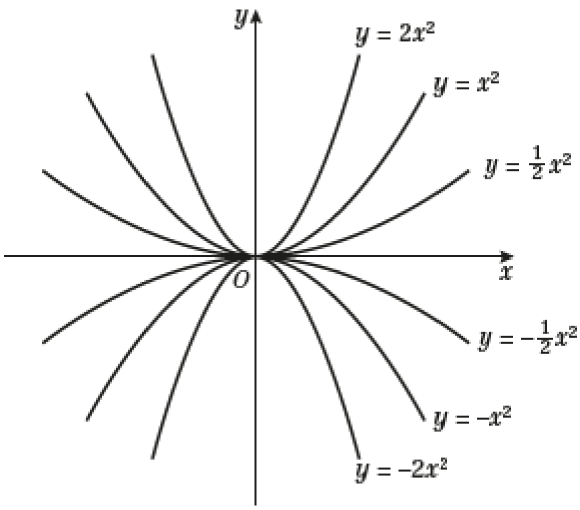
$\Rightarrow \ln y = \ln x^2 + \ln A$ using the power law

$\Rightarrow \ln y = \ln Ax^2$ using the multiplication law

$\Rightarrow y = Ax^2$

The family of solution curves are parabola.

Sketching the solution curves for $A = -2, -1, -\frac{1}{2}, \frac{1}{2}, 1, 2$ gives:



1 d Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{x}{y}$$

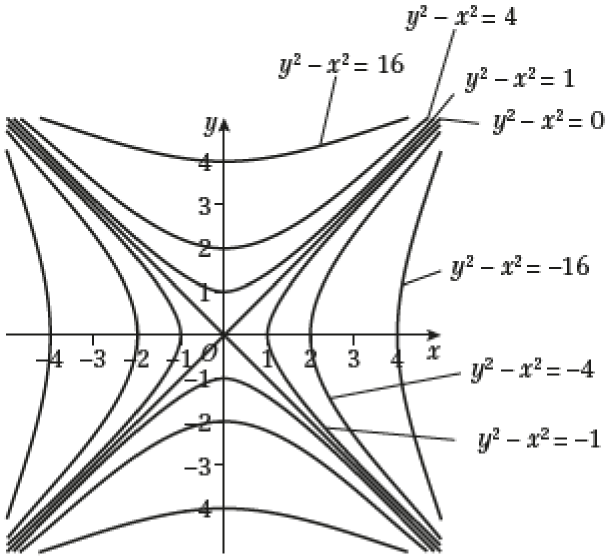
$$\Rightarrow \int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c \quad \text{or} \quad y^2 - x^2 = 2c$$

$y^2 - x^2 = 0 \Rightarrow (y - x)(y + x) = 0$, and the graph of this equation are straight lines $y = x$ and $y = -x$

$y^2 - x^2 = 2c$ for $c \neq 0$ is a hyperbola with asymptotes $y = x$ and $y = -x$

Sketching some of the solution curves gives:



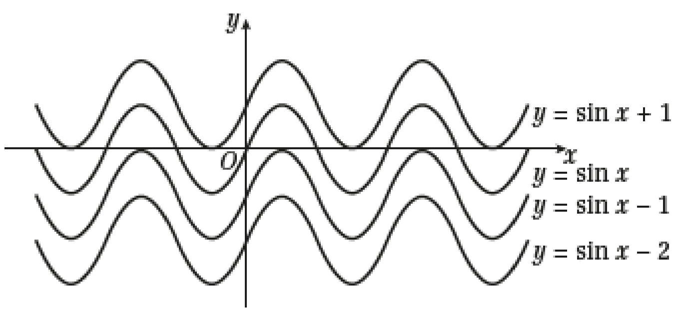
e $\frac{dy}{dx} = \cos x$

$\Rightarrow y = \sin x + c$

The family of solution curves are sin curves.

The graph of $y = \sin x + c$ is a translation of $y = \sin x$ by the vector $\begin{pmatrix} 0 \\ c \end{pmatrix}$

Sketching some of the solution curves gives:



1 f Separating the variables and integrating:

$$\frac{dy}{dx} = y\cot x \quad 0 < x < \pi$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{\cos x}{\sin x}\,dx$$

$$\Rightarrow \ln|y| = \ln|\sin x| + c \qquad \text{integrating } \frac{\cos x}{\sin x} \text{ using the reverse chain rule}$$

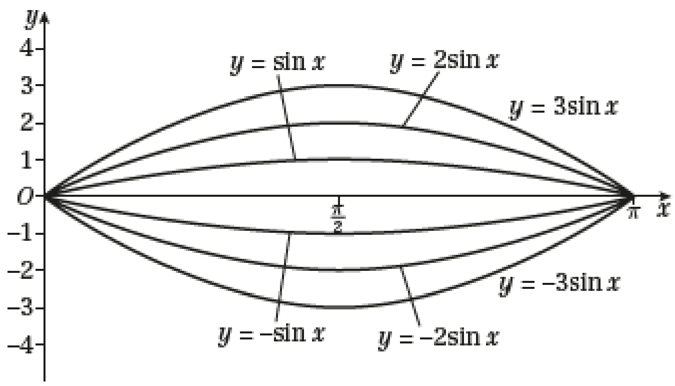
Expressing the constant as $\ln|A|$ and simplifying using the laws of logarithms:

$$\ln|y| = \ln|\sin x| + \ln|A|$$

$$\Rightarrow \ln|y| = \ln|A\sin x|$$

$$\Rightarrow y = A\sin x$$

The family of solution curves are sin curves for $0 < x < \pi$ with varying amplitudes.
Sketching some of the solution curves gives:



2 **a** Separating the variables and integrating:

$$\frac{dy}{dx} = \frac{-xy}{9-x^2}$$

$$\Rightarrow \int \frac{1}{y}\,dy = -\int \frac{x}{9-x^2}\,dx$$

$$\Rightarrow \ln y = \frac{1}{2}\ln(9-x^2) + \ln A$$

$$\Rightarrow 2\ln y = \ln A^2(9-x^2)$$

$$\Rightarrow \ln y^2 = \ln A^2(9-x^2)$$

$$\Rightarrow y^2 = 9A^2 - A^2x^2$$

Let $A^2 = k$, so be definition k is a positive constant

Then $y^2 + kx^2 = 9k$

b If the solution passes through (2, 5) then

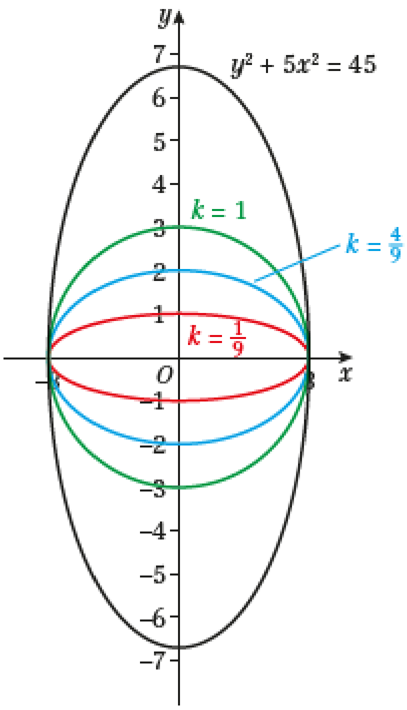
$25 + 4k = 9k$

$25 = 5k \Rightarrow k = 5$

So the equation is $y^2 + 5x^2 = 45$

c The solution curves are all ellipses, except when $k = 1$ when the curve is circle.

When $y = 0$, $x = \pm 3$, when $x = 0$, $y = \pm\sqrt{9k}$



3 a $x\dfrac{dy}{dx}+y=\cos x$

$\Rightarrow \dfrac{d}{dx}(xy)=\cos x$ as $\dfrac{d}{dx}(xy)=x\dfrac{dy}{dx}+y$ from the product rule

$\Rightarrow xy=\int \cos dx=\sin x+c$

So $y=\dfrac{1}{x}\sin x+\dfrac{c}{x}$

b $e^{-x}\dfrac{dy}{dx}-e^{-x}y=xe^{x}$

$\Rightarrow \dfrac{d}{dx}(e^{-x}y)=xe^{x}$ as $\dfrac{d}{dx}(e^{-x}y)=e^{-x}\dfrac{dy}{dx}-e^{-x}y$ from the product rule

$\Rightarrow e^{-x}y=\int xe^{x}dx=xe^{x}-\int e^{x}dx$ using integration by parts formula (with $u=x$, $v=e^{x}$)

$=xe^{x}-e^{x}+c$

So $y=xe^{2x}-e^{2x}+ce^{x}$ multiplying both sides by e^{x}

c $\sin x\dfrac{dy}{dx}+y\cos x=3$

$\Rightarrow \dfrac{d}{dx}(y\sin x)=3$ as $\dfrac{d}{dx}(y\sin x)=\sin x\dfrac{dy}{dx}+y\cos x$ from the product rule

$\Rightarrow y\sin x=\int 3\ dx$

$\Rightarrow y\sin x=3x+c$

So $y=\dfrac{3x}{\sin x}+\dfrac{c}{\sin x}=3x\operatorname{cosec} x+c\operatorname{cosec} x$

d $\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{1}{x^2}y=e^{x}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{x}y\right)=e^{x}$ as $\dfrac{d}{dx}\left(\dfrac{1}{x}y\right)=\dfrac{1}{x}\dfrac{dy}{dx}-\dfrac{1}{x^2}y$ from the product rule

$\Rightarrow \dfrac{1}{x}y=\int e^{x}\,dx=e^{x}+c$

So $y=xe^{x}+cx$

3 e Simplify the left-hand side by noting that from the product and chain rules

$$\frac{d}{dx}(x^2e^y) = x^2\frac{d(e^y)}{dx} + 2xe^y \qquad \text{the product rule}$$

$$= x^2\frac{d(e^y)}{dy}\frac{dy}{dx} + 2xe^y \qquad \text{the chain rule}$$

$$= x^2e^y\frac{dy}{dx} + 2xe^y$$

So $x^2e^y\frac{dy}{dx} + 2xe^y = x \Rightarrow \frac{d}{dx}(x^2e^y) = x$

$\Rightarrow x^2e^y = \int x\,dx = \frac{x^2}{2} + c$

$\Rightarrow e^y = \frac{1}{2} + \frac{c}{x^2}$

So $y = \ln\left(\frac{1}{2} + \frac{c}{x^2}\right)$

f $4xy\frac{dy}{dx} + 2y^2 = x^2$

$\Rightarrow \frac{d}{dx}(2xy^2) = x^2$ using the product and chain rules

$\Rightarrow 2xy^2 = \int x^2\,dx = \frac{1}{3}x^3 + c$

$\Rightarrow y^2 = \frac{1}{6}x^2 + \frac{c}{2x}$

So $y = \pm\sqrt{\frac{1}{6}x^2 + \frac{c}{2x}}$

4 a The equation is in the form $\frac{dy}{dx} + P(x)y = Q(x)$, so the integrating factor is $e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$

Multiplying the equation by this factor gives:

$$e^{x^2}\frac{dy}{dx} + e^{x^2}2xy = 1$$

$$\Rightarrow \frac{d}{dx}(ye^{x^2}) = 1$$

$$\Rightarrow ye^{x^2} = \int 1 dx = x + c$$

So $y = \frac{x+c}{e^{x^2}} = xe^{-x^2} + ce^{-x^2}$

b As $x \to \infty$, e^{x^2} becomes much larger than x; therefore, $y \to 0$.

5 a $x^2\frac{dy}{dx} + 2xy = 2x + 1$

$$\Rightarrow \frac{d}{dx}(x^2y) = 2x + 1$$

$$\Rightarrow x^2y = \int (2x+1) dx$$

$$\Rightarrow x^2y = x^2 + x + c$$

So $y = 1 + \frac{1}{x} + \frac{c}{x^2}$

5 b When $x = -\frac{1}{2}$, $y = 0$, $1 - 2 + 4c = 0 \Rightarrow 4c = 1 \Rightarrow c = \frac{1}{4}$

So $y = 1 + \frac{1}{x} + \frac{1}{4x^2}$

When $x = -\frac{1}{2}$, $y = 3$, $1 - 2 + 4c = 3 \Rightarrow 4c = 4 \Rightarrow c = 1$

So $y = 1 + \frac{1}{x} + \frac{1}{x^2}$

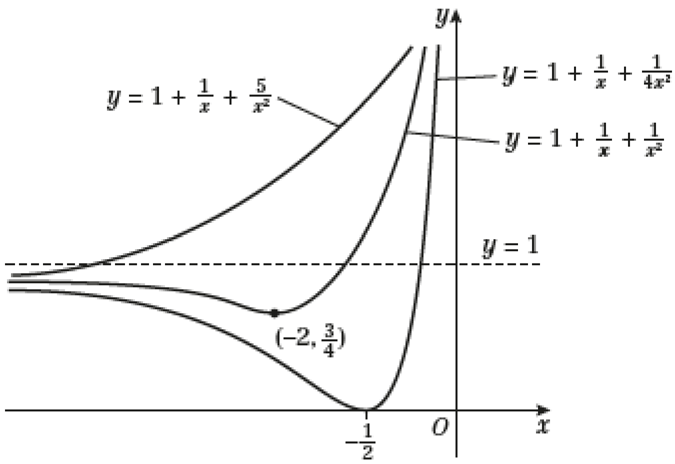
When $x = -\frac{1}{2}$, $y = 19$, $1 - 2 + 4c = 19 \Rightarrow 4c = 20 \Rightarrow c = 5$

So $y = 1 + \frac{1}{x} + \frac{5}{x^2}$

The curves have a horizontal asymptote at $y = 1$ and a vertical asymptote at $x = 0$

When $y = 1$, $\frac{1}{x} + \frac{c}{x^2} = 0 \Rightarrow x = -c$. When $y = 0$, $x^2 + x + c = 0$. There are no real roots for $c > \frac{1}{4}$.

So a sketch of the three curves for $x < 0$ is



6 a $\ln x \frac{dy}{dx} + \frac{y}{x} = \frac{1}{(x+1)(x+2)}$

$\Rightarrow \frac{d}{dx}(y \ln x) = \frac{1}{(x+1)(x+2)}$ as $\frac{d}{dx}(y \ln x) = \ln x \frac{dy}{dx} + \frac{y}{x}$ using the product rule

$\Rightarrow y \ln x = \int \frac{1}{(x+1)(x+2)} dx$

$= \int \left(\frac{(x+2)-(x+1)}{(x+1)(x+2)} \right) dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx$

$= \ln(x+1) - \ln(x+2) + \ln A$

So $y = \frac{\ln(x+1) - \ln(x+2) + \ln A}{\ln x} = \frac{\ln \frac{A(x+1)}{(x+2)}}{\ln x}$

6 b When $x = 2,\ y = 2,\ 2 = \dfrac{\ln \frac{3}{4} A}{\ln 2}$

So $\ln \dfrac{3}{4} A = 2\ln 2 = \ln 4$

$\Rightarrow \dfrac{3}{4} A = 4 \Rightarrow A = \dfrac{16}{3}$

So the solution is $y = \dfrac{\ln \dfrac{16(x+1)}{3(x+2)}}{\ln x}$

7 a The integrating factor is $e^{\int 2\,dx} = e^{2x}$

Multiplying the equation by this factor gives:

$e^{2x} \dfrac{dy}{dx} + 2e^{2x} y = e^{2x} e^{x}$

$\Rightarrow \dfrac{d}{dx}(e^{2x} y) = e^{3x}$

$\Rightarrow e^{2x} y = \int e^{3x}\,dx = \dfrac{1}{3} e^{3x} + c$

So $y = \dfrac{1}{3} e^{x} + ce^{-2x}$

b The integrating factor is $e^{\int \cot x\,dx} = e^{\ln \sin x} = \sin x$

Multiplying the equation by this factor gives:

$\sin x \dfrac{dy}{dx} + y \cos x = \sin x$

$\Rightarrow \dfrac{d}{dx}(y \sin x) = \sin x$

$\Rightarrow y \sin x = \int \sin x\,dx = -\cos x + c$

So $y = -\cot x + c\,\mathrm{cosec}\,x$

c The integrating factor is $e^{\int \sin x\,dx} = e^{-\cos x}$

Multiplying the equation by this factor gives:

$e^{-\cos x} \dfrac{dy}{dx} + y \sin x e^{-\cos x} = e^{-\cos x} e^{\cos x}$

$\Rightarrow \dfrac{d}{dx}(y e^{-\cos x}) = 1$

$\Rightarrow y e^{-\cos x} = x + c$

So $y = x e^{\cos x} + c e^{\cos x}$

7 **d** The integrating factor is $e^{\int -1\,dx} = e^{-x}$

> Note that $P(x) = -1$ and the minus sign is important.

Multiplying the equation by this factor gives:

$$e^{-x}\frac{dy}{dx} - ye^{-x} = e^{2x}e^{-x}$$

$$\Rightarrow \frac{d}{dx}(ye^{-x}) = e^{x}$$

$$\Rightarrow ye^{-x} = \int e^{x}\,dx = e^{x} + c$$

So $y = e^{2x} + ce^{x}$

e The integrating factor is $e^{\int \tan x\,dx} = e^{\ln\sec x} = \sec x$

Multiplying the equation by this factor gives:

$$\sec x\frac{dy}{dx} + y\sec x\tan x = x\cos x\sec x$$

$$\Rightarrow \frac{d}{dx}(y\sec x) = x$$

$$\Rightarrow y\sec x = \int x\,dx = \frac{1}{2}x^2 + c$$

So $y = \left(\frac{1}{2}x^2 + c\right)\cos x$

f The integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiplying the equation by this factor gives:

$$x\frac{dy}{dx} + y = \frac{x}{x^2}$$

$$\Rightarrow \frac{d}{dx}(xy) = \frac{1}{x}$$

$$\Rightarrow xy = \int \frac{1}{x}dx = \ln x + c$$

So $y = \frac{1}{x}\ln x + \frac{c}{x}$

7 **g** Divide both sides by x^2 to get an equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=Q(x)$. This gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x}y=\frac{x}{x+2} \qquad \textbf{(1)}$$

The integrating factor is $\mathrm{e}^{\int -\frac{1}{x}\mathrm{d}x}=\mathrm{e}^{-\ln x}=\mathrm{e}^{\ln\frac{1}{x}}=\frac{1}{x}$

Multiplying equation (**1**) by this factor gives:

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x^2}y=\frac{1}{x+2}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right)=\frac{1}{x+2}$$

$$\Rightarrow \frac{1}{x}y=\int\frac{1}{x+2}\mathrm{d}x=\ln(x+2)+c$$

So $y=x\ln(x+2)+cx$

h Divide both sides by x^2 to get an equation in the form $\frac{\mathrm{d}y}{\mathrm{d}x}+P(x)y=Q(x)$. This gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{3x}y=\frac{1}{3} \qquad \textbf{(1)}$$

The integrating factor is $\mathrm{e}^{\int\frac{1}{3x}\mathrm{d}x}=\mathrm{e}^{\frac{1}{3}\ln x}=\mathrm{e}^{\ln x^{\frac{1}{3}}}=x^{\frac{1}{3}}$

Multiplying equation (**1**) by $x^{\frac{1}{3}}$ gives:

$$x^{\frac{1}{3}}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{3}x^{-\frac{2}{3}}y=\frac{1}{3}x^{\frac{1}{3}}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{3}}y\right)=\frac{1}{3}x^{\frac{1}{3}}$$

$$\Rightarrow x^{\frac{1}{3}}y=\int\frac{1}{3}x^{\frac{1}{3}}\mathrm{d}x=\frac{1}{4}x^{\frac{4}{3}}+c$$

So $y=\frac{1}{4}x+cx^{-\frac{1}{3}}$

7 i Dividing both sides by $(x + 2)$ gives:

$$\frac{dy}{dx}-\frac{1}{(x+2)}y=1 \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \frac{-1}{(x+2)}dx}=e^{-\ln(x+2)}=e^{\ln\frac{1}{x+2}}=\frac{1}{x+2}$

Multiplying equation (**1**) by the integrating factor:

$$\frac{1}{(x+2)}\frac{dy}{dx}-\frac{1}{(x+2)^2}y=\frac{1}{(x+2)}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{(x+2)}y\right]=\frac{1}{x+2}$$

$$\Rightarrow \frac{1}{(x+2)}y=\int\frac{1}{x+2}\,dx=\ln(x+2)+c$$

So $y=(x+2)\ln(x+2)+c(x+2)$

j Dividing both sides by x gives:

$$\frac{dy}{dx}+\frac{4}{x}y=\frac{e^x}{x^3} \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \frac{4}{x}dx}=e^{4\ln x}=e^{\ln x^4}=x^4$

Multiplying equation (**1**) by the integrating factor:

$$x^4\frac{dy}{dx}+4x^3y=xe^x$$

$$\Rightarrow \frac{d}{dx}(x^4y)=xe^x$$

$$\Rightarrow x^4y=\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$$

Integrating xe^x using integration by parts

So $y=\frac{1}{x^3}e^x-\frac{1}{x^4}e^x+\frac{c}{x^4}$

8 Dividing both sides by x gives:

$$\frac{dy}{dx}+\frac{2}{x}y=\frac{1}{x}e^x \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

Multiplying equation (**1**) by x^2

$$x^2\frac{dy}{dx}+2xy=xe^x$$

$$\Rightarrow \frac{d}{dx}(x^2y)=xe^x$$

$$\Rightarrow x^2y=\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$$

So $y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{c}{x^2}$

Given that $y = 1$ when $x = 1$, then $1=e-e+c\Rightarrow c=1$

So the required equation is $y=\frac{1}{x}e^x-\frac{1}{x^2}e^x+\frac{1}{x^2}$

9 Dividing both sides by x^3 gives:

$$\frac{dy}{dx}-\frac{1}{x}y=\frac{1}{x^3} \qquad (\mathbf{1})$$

The integrating factor is $e^{-\int \frac{1}{x}dx}=e^{-\ln x}=e^{\ln\frac{1}{x}}=\frac{1}{x}$

Multiplying equation (**1**) by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=\frac{1}{x^4}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right)=\frac{1}{x^4}$$

$$\frac{1}{x}y=\int\frac{1}{x^4}dx=\int x^{-4}\,dx=-\frac{1}{3}x^{-3}+c$$

So $y=-\frac{1}{3}x^{-2}+cx=-\frac{1}{3x^2}+cx$

But $y=1$ when $x=1$, so $1=-\frac{1}{3}+c\Rightarrow c=\frac{4}{3}$

So the required equation is $y=-\frac{1}{3x^2}+\frac{4x}{3}$

10 a Dividing both sides by $\left(x+\frac{1}{x}\right)$ gives:

$\frac{dy}{dx}+\frac{2}{\left(x+\frac{1}{x}\right)}y=\frac{2(x^2+1)^2}{\left(x+\frac{1}{x}\right)}$, which simplifies to

$$\frac{dy}{dx}+\frac{2x}{x^2+1}\times y=2x(x^2+1) \qquad (\mathbf{1})$$

The integrating factor is $e^{\int\frac{2x}{x^2+1}dx}=e^{\ln(x^2+1)}=(x^2+1)$

Multiplying equation (**1**) by (x^2+1)

$$(x^2+1)\frac{dy}{dx}+2xy=2x(x^2+1)^2$$

$$\Rightarrow \frac{d}{dx}\left((x^2+1)y\right)=2x(x^2+1)^2$$

$$y(x^2+1)=\int 2x(x^2+1)^2\,dx=\frac{1}{3}(x^2+1)^3+c$$

So $y=\frac{1}{3}(x^2+1)^2+\frac{c}{(x^2+1)}$

b Given that $y = 1$ when $x = 1$, then $1=\frac{1}{3}\times 4+\frac{1}{2}c\Rightarrow c=-\frac{2}{3}$

So the required equation is $y=\frac{1}{3}(x^2+1)^2-\frac{2}{3(x^2+1)}$

11 a Dividing both sides by $\cos x$ gives:

$$\frac{dy}{dx} + y\sec x = \sec x \qquad \textbf{(1)}$$

Using the standard result $\int \sec x\,dx = \ln(\sec x + \tan x)$, (you will not be expected to prove this result)the integrating factor is

$$e^{\int \sec x\,dx} = e^{\ln(\sec x + \tan x)} = \sec x + \tan x$$

Multiplying equation (**1**) by this factor gives:

$$(\sec x + \tan x)\frac{dy}{dx} + (\sec^2 x + \sec x \tan x)y = \sec^2 x + \sec x \tan x$$

$$\Rightarrow \frac{d}{dx}\big((\sec x + \tan x)y\big) = \sec^2 x + \sec x \tan x$$

$$\Rightarrow (\sec x + \tan x)y = \int \sec^2 x + \sec x \tan x\,dx = \tan x + \sec x + c$$

So $y = 1 + \dfrac{c}{\sec x + \tan x}$

b Given that $y = 2$ when $x = 0$, then $2 = 1 + \dfrac{c}{1+0} \Rightarrow c = 1$

So $y = 1 + \dfrac{1}{\sec x + \tan x}$

Dividing top and bottom by $\cos x$ gives $y = 1 + \dfrac{\cos x}{1 + \sin x}$

12 Rewriting the equation as $\dfrac{1}{\sqrt{y^2 - 4}}dy = x\,dx$

Integrate the left-hand side using the substitution $y = 2\cosh u$

$\dfrac{dy}{du} = 2\sinh u$, so dy can be replaced by $2\sinh u\,du$

$$\int \frac{1}{\sqrt{y^2-4}}dy = \int \frac{1}{\sqrt{4\cosh^2 u - 4}} 2\sinh u\,du = \int \frac{1}{2\sinh u} 2\sinh u\,du = \int 1\,du$$

$$= u + c = \operatorname{arcosh}\left(\frac{y}{2}\right) + c$$

So integrating both sides of the rewritten equation gives:

$$\operatorname{arcosh}\left(\frac{y}{2}\right) = \frac{x^2}{2} + c \Rightarrow \frac{y}{2} = \cosh\left(\frac{x^2}{2} + c\right)$$

So the general solution is $y = 2\cosh\left(\dfrac{x^2}{2} + c\right)$

13 a Rewriting the equation as $\dfrac{1}{y}dy = \cosh x\,dx$

Integrating both sides gives

$\ln y = \sinh x + c$ and therefore the general solution is $y = e^{\sinh x + c}$

b Given that $y = e$ when $x = 0$, then $e = e^{\sinh 0 + c} = e^c \Rightarrow c = 1$. So the particular solution is $y = e^{\sinh x + 1}$

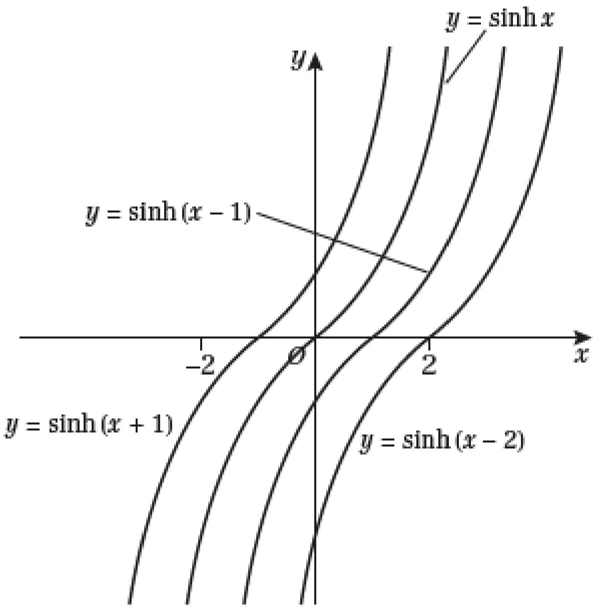
14 a Rewriting the equation as $\frac{1}{\sqrt{1+y^2}}\mathrm{d}y = \mathrm{d}x$

The left-hand side is the derivative of arsinh y, so integrating both sides gives:

$\text{arsinh}\, y = x + c \Rightarrow y = \sinh(x+c)$

b The graph of $y = \sinh(x+c)$ is a translation of $y = \sinh x$ by the vector $\begin{pmatrix} -c \\ 0 \end{pmatrix}$

This is a sketch of some solution curves.



15 a Dividing both sides by cos x gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sec x \qquad \textbf{(1)}$$

The integrating factor is $e^{\int \tan x \, \mathrm{d}x} = e^{\ln|\sec x|} = \sec x$

Multiply both sides of equation (**1**) by sec x

$$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \sec^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(y\sec x) = \sec^2 x$$

$$\Rightarrow y\sec x = \int \sec^2 x \, \mathrm{d}x$$

$$\Rightarrow y\sec x = \tan x + c$$

$$y = \sin x + c\cos x$$

b Given that $y = 3$ when $x = \pi$, then $3 = \sin\pi + c\cos\pi = 0 - c \Rightarrow c = -3$.

So the particular solution is $y = \sin x - 3\cos x$

15 c If $x=\frac{\pi}{2}$, then $y=\sin\frac{\pi}{2}+c\cos\frac{\pi}{2}=1+c\times 0=1$ for any value of c

Similarly if $x=\frac{3\pi}{2}$, then $y=\sin\frac{3\pi}{2}+c\cos\frac{3\pi}{2}=-1+c\times 0=-1$ for any value of c

So $\left(\frac{\pi}{2},1\right)$ and $\left(\frac{3\pi}{2},-1\right)$ lie on all solution curves.

16 Dividing by a gives $\frac{dy}{dx}+\frac{b}{a}y=0$

The integrating factor is $e^{\int\frac{b}{a}dx}=e^{\frac{bx}{a}}$

Multiplying by this factor gives

$e^{\frac{bx}{a}}\frac{dy}{dx}+\frac{b}{a}e^{\frac{bx}{a}}y=0$

$\Rightarrow \frac{d}{dx}\left(e^{\frac{bx}{a}}y\right)=0$

$\Rightarrow e^{\frac{bx}{a}}y=c$

So the general solution is $y=ce^{-\frac{bx}{a}}$