## Hyperbolic Functions 6D

**1 a** $\frac{d}{dx}(\sinh 2x) = 2\cosh 2x$

**b** $\frac{d}{dx}(\cosh 5x) = 5\sinh 5x$

**c** $\frac{d}{dx}(\tanh 2x) = 2\text{sech}^2 2x$

**d** $\frac{d}{dx}(\sinh 3x) = 3\cosh 3x$

**e** $\frac{d}{dx}(\coth 4x) = -4\text{cosech}^2 4x$

**f** $\frac{d}{dx}(\text{sech}\, 2x) = \frac{-1}{(\cosh 2x)^2} \times 2\sinh 2x$

$= -2\frac{\sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x}$

$= -2\tanh 2x\ \text{sech} 2x$

**g** $\frac{d}{dx}(e^{-x}\sinh x) = -e^{-x}\sinh x + e^{-x}\cosh x$

$= e^{-x}(\cosh x - \sinh x)$

**h** $\frac{d}{dx}(x\cosh 3x) = \cosh 3x + 3x\sinh 3x$

**i** $\frac{d}{dx}\left(\frac{\sinh x}{3x}\right) = \frac{\cosh x}{3x} - \frac{\sinh x}{3x^2}$

$= \frac{x\cosh x - \sinh x}{3x^2}$

**j** $\frac{d}{dx}(x^2\cosh 3x) = 2x\cosh 3x + x^2 \times 3\sinh 3x$

$= x(2\cosh 3x + 3x\sinh 3x)$

**k** $\frac{d}{dx}(\sinh 2x\cosh 3x) = 2\cosh 2x\cosh 3x + \sinh 2x \times 3\sinh 3x$

$= 2\cosh 2x\cosh 3x + 3\sinh 2x\sinh 3x$

**l** $\frac{d}{dx}(\ln\cosh x) = \frac{1}{\cosh x} \times \sinh x$

$= \tanh x$

**1 m** $\frac{d}{dx}(\sinh x^3) = 3x^2 \cosh x^3$

**n** $\frac{d}{dx}(\cosh^2 2x) = (2\cosh 2x)(2\sinh 2x)$
$= 4\cosh 2x \sinh 2x$

**o** $\frac{d}{dx}(e^{\cosh x}) = \sinh x e^{\cosh x}$

**p** $\frac{d}{dx}(\text{cosech}x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{0 - 1 \times \cosh x}{\sinh^2 x}$
$= -\coth x \,\text{cosech}x$

**2** $y = a\cosh nx + b\sinh nx$
Differentiate with respect to $x$
$\frac{dy}{dx} = an\sinh nx + nb\cosh nx$
$\frac{d^2y}{dx^2} = an^2\cosh nx + bn^2\sinh nx$
$= n^2(a\cosh nx + b\sinh nx)$
$\frac{d^2y}{dx^2} = n^2 y$

**3** To find the stationary point of the curve $y = 12\cosh x - \sinh x$, we differentiate and set equal to 0.
$\frac{dy}{dx} = 12\sinh x - \cosh x = 0$
$\tanh x = \frac{1}{12}$
$x = \text{artanh}\left(\frac{1}{12}\right).$
Using the formula $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, we find that
$x = \frac{1}{2}\ln\left(\frac{1+\frac{1}{12}}{1-\frac{1}{12}}\right)$
$= \frac{1}{2}\ln\left(\frac{13}{11}\right)$

**3** Substituting this value for $x$ back into the given equation for $y$, we obtain

$$y = 12\cosh\left(\frac{1}{2}\ln\left(\frac{13}{11}\right)\right) - \sinh\left(\frac{1}{2}\ln\left(\frac{13}{11}\right)\right)$$

$$= \frac{12}{2}\left(\sqrt{\frac{13}{11}} + \sqrt{\frac{11}{13}}\right) - \frac{1}{2}\left(\sqrt{\frac{13}{11}} - \sqrt{\frac{11}{13}}\right)$$

$$= \frac{1}{2}\left(11\sqrt{\frac{13}{11}} + 13\sqrt{\frac{11}{13}}\right)$$

$$= \sqrt{143}.$$

So the coordinates of the stationary point are $\left(\frac{1}{2}\ln\left(\frac{13}{11}\right), \sqrt{143}\right)$.

**4** $$y = \cosh 3x \sinh x$$

$$\frac{dy}{dx} = 3\sinh 3x \sinh x + \cosh 3x \cosh x$$

$$\frac{d^2y}{dx^2} = 9\cosh 3x \sinh x + 3\sinh 3x \cosh x + 3\sinh 3x \cosh x + \cosh 3x \sinh x$$

$$= 10\cosh 3x \sinh x + 6\sinh 3x \cosh x$$

$$= 2(5\cosh 3x \sinh x + 3\sinh 3x \cosh x)$$

**5 a** Let $y = \text{arcosh}2x$ then $\cosh y = 2x$

Differentiate with respect to $x$

$$\sinh y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sinh y}$$

$$= \frac{2}{\sqrt{\cosh^2 y - 1}} \text{ but } \cosh y = 2x$$

$$\text{so } \frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}$$

**b** Let $y = \text{arsinh}(x+1)$ then $\sinh y = x + 1$

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{\sinh^2 y + 1}} \text{ but } \sinh y = x + 1$$

$$\text{so } \frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2 + 1}}$$

**5 c** Let $y = \text{artanh}3x$

$\tanh y = 3x$

$\text{sech}^2 y \dfrac{dy}{dx} = 3$

$\dfrac{dy}{dx} = \dfrac{3}{\text{sech}^2 y}$

$\dfrac{dy}{dx} = \dfrac{3}{1 - \tanh^2 y}$

$\dfrac{dy}{dx} = \dfrac{3}{1 - 9x^2}$

**d** Let $y = \text{arsech}x$

$\text{sech}y = x$

$\dfrac{1}{\cosh y} = x$

$1 = x\cosh y$

Differentiate with respect to $x$

$0 = \cosh y + x\sinh y \dfrac{dy}{dx}$

$x\sinh y \dfrac{dy}{dx} = -\cosh y$

$\dfrac{dy}{dx} = \dfrac{-\cosh y}{x\sinh y}$

$= \dfrac{-1}{x\tanh y}$

$= \dfrac{-1}{x(1 - \text{sech}^2 y)^{\frac{1}{2}}}$

$= \dfrac{-1}{x(1 - x^2)^{\frac{1}{2}}}$

$= -\dfrac{1}{x^2\sqrt{\dfrac{1}{x^2} - 1}}$

**e** Let $y = \text{arcosh}x^2$

Let $t = x^2 \quad y = \text{arcosh}t$

$\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right)\left(\dfrac{dt}{dx}\right) = \left(\dfrac{1}{\sqrt{t^2 - 1}}\right)(2x)$

$\dfrac{dy}{dx} = \dfrac{2x}{\sqrt{x^4 - 1}}$

**5 f** $y = \text{arcosh}\,3x$

Let $t = 3x \quad y = \text{arcosh}t$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \left(\frac{1}{\sqrt{t^2-1}}\right)(3)$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2-1}}$$

**g** $y = x^2\text{arcosh}x$

$$\frac{dy}{dx} = 2x\text{arcosh}x + \frac{x^2}{\sqrt{x^2-1}}$$

**h** $y = \text{arsinh}\,\frac{x}{2}$

Let $t = \frac{x}{2} \quad y = \text{arsinh}t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{1}{\sqrt{t^2+1}}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2\sqrt{\left(\frac{x}{2}\right)^2+1}} = \frac{1}{\sqrt{x^2+4}}$$

**i** $y = e^{x^3}\text{arsinh}x$

$$\frac{dy}{dx} = 3x^2e^{x^3}\text{arsinh}x + \frac{e^{x^3}}{\sqrt{x^2+1}}$$

**j** $y = \text{arsinh}x\text{arcosh}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\text{arcosh}x + \frac{1}{\sqrt{x^2-1}}\text{arsinh}x$$

**k** $y = \text{arcosh}x\text{sech}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}\text{sech}x - \text{arcosh}x \tanh x\text{sech}x$$

$$= \text{sech}x\left(\frac{1}{\sqrt{x^2-1}} - \text{arcosh}x \tanh x\right)$$

**l** $y = x\text{arcosh}3x$

$$\frac{dy}{dx} = \text{arcosh}3x + x \times \frac{3}{\sqrt{9x^2-1}}$$

$$\frac{dy}{dx} = \text{arcosh}3x + \frac{3x}{\sqrt{9x^2-1}}$$

**6 a** $y = \text{arcosh}x$

$\cosh y = x$

$\sinh y \dfrac{dy}{dx} = 1 \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}}$

but $\cosh y = x$ so

$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - 1}}$

**b** $y = \text{artanh}x$

$\tanh y = x$

$\text{sech}^2 y \dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\text{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y}$

but $\tanh y = x$ so

$\dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$

**7** $y = \text{artanh}\dfrac{e^x}{2}$

Let $t = \dfrac{e^x}{2}$ $y = \text{artanh}t$

$\dfrac{dt}{dx} = \dfrac{e^x}{2}$ $\dfrac{dy}{dt} = \dfrac{1}{1 - t^2}$

Then $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right)\left(\dfrac{dt}{dx}\right) = \dfrac{1}{1 - t^2} \times \dfrac{e^x}{2}$

$= \dfrac{1}{1 - \left(\frac{e^x}{2}\right)^2} \times \dfrac{e^x}{2}$

$= \dfrac{\frac{e^x}{2}}{\frac{4 - e^{2x}}{4}}$

$\dfrac{dy}{dx} = \dfrac{2e^x}{4 - e^{2x}}$

$(4 - e^{2x})\dfrac{dy}{dx} = 2e^x$

**8** $y = \text{arsinh}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} = (x^2+1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2+1)^{-\frac{3}{2}}2x$$

$$= \frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{-1(x^2+1)^{\frac{3}{2}} - \frac{3}{2}(x^2+1)^{\frac{1}{2}} \times 2x \times -x}{(x^2+1)^3}$$

$$= \frac{3x^2(x^2+1)^{\frac{1}{2}} - (x^2+1)^{\frac{3}{2}}}{(x^2+1)^3}$$

$$= \frac{3x^2}{(x^2+1)^{\frac{5}{2}}} - \frac{1}{(x^2+1)^{\frac{3}{2}}}$$

$$(x^2+1)\frac{d^3y}{dx^3} = \frac{3x^2}{(x^2+1)^{\frac{3}{2}}} - \frac{1}{(x^2+1)^{\frac{1}{2}}}$$

$$= -3x\frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

**9** $y = (\text{arcosh}x)^2$

$$\frac{dy}{dx} = 2\text{arcosh}x \times \frac{1}{\sqrt{x^2-1}}$$

$$= 2(x^2-1)^{-\frac{1}{2}}\text{arcosh}x$$

$$\frac{d^2y}{dx^2} = -(x^2-1)^{-\frac{3}{2}}2x\text{arcosh}x + 2(x^2-1)^{-\frac{1}{2}} \times \frac{1}{\sqrt{x^2-1}}$$

$$= \frac{-2x\text{arcosh}x}{(x^2-1)^{\frac{3}{2}}} + \frac{2}{x^2-1}$$

**10** $y = \text{artanh}x \quad x = \frac{12}{13} \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln 25 = \ln 5$

$$\frac{dy}{dx} = \frac{1}{1-x^2} = \frac{1}{1-\left(\frac{12}{13}\right)^2} = \frac{169}{25}$$

Tangent is

$$(y - \ln 5) = \frac{169}{25}\left(x - \frac{12}{13}\right)$$

$$25y - 25\ln 5 = 169x - 156$$

**11** In order to find the normal to the curve $y = \text{arcosh}\, 2x$ at the point $x = 2$, we first must find the value of $\frac{dy}{dx}$ at that point.

$$\frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}\text{, so at } x = 2,$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{4 \times 2^2 - 1}}$$

$$= \frac{2}{\sqrt{15}}.$$

The gradient of the normal at this point is therefore $-\frac{\sqrt{15}}{2}$

The $y$ value at $x = 2$ is $\text{arcosh}\, 4 = \ln\left(4 + \sqrt{15}\right)$.

So now we substitute our values for $x$, $y$, $m$ into $y = mx + c$ in order to find $c$.

$$y = mx + c$$

$$\ln\left(4 + \sqrt{15}\right) = -\frac{\sqrt{15}}{2} \times 2 + c$$

$$c = \ln\left(4 + \sqrt{15}\right) + \sqrt{15}.$$

So we have $y = -\frac{\sqrt{15}}{2}x + \sqrt{15} + \ln\left(4 + \sqrt{15}\right)$.

**12 a** We differentiate and evaluate at 0 until we have 3 non-zero terms.

$$f(x) = \cosh x \Rightarrow f(0) = 1$$

$$f'(x) = \sinh x \Rightarrow f'(0) = 0$$

$$f''(x) = \cosh x \Rightarrow f''(0) = 1$$

$$f'''(x) = \sinh x \Rightarrow f'''(0) = 0$$

$$f^{(4)}(x) = \cosh x \Rightarrow f^{(4)}(0) = 1.$$

Now we use the standard Maclaurin series expansion and obtain

$$\cosh x \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{24}.$$

**b** Using the approximation, $\cosh 0.2 \approx 1 + \frac{0.2^2}{2!} + \frac{0.2^4}{4!} = 1.020067(6 \text{ d.p.})$.

$$error = \left|\frac{1.020066667 - \cosh 0.2}{\cosh 0.2}\right| \times 100$$

$$= 8.7 \times 10^{-6}\%.$$

**13 a** We differentiate and evaluate at 0 until we have 3 non-zero terms.

$f(x) = \sinh x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = 1$

$f''(x) = \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = 1$

$f^{(4)}(x) = \sinh x \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = \cosh x \Rightarrow f^{(5)}(0) = 1.$

Now we use the standard Maclaurin series expansion and obtain

$$\sinh x \approx x + \frac{x^3}{3!} + \frac{x^5}{5!}$$
$$= x + \frac{x^3}{6} + \frac{x^5}{120}.$$

**b** Since for all integers $n \geq 1$,
$$f^{(2n-2)}(x) = \sinh x \Rightarrow f^{(2n-2)}(0) = 0,$$
$$f^{(2n-1)}(x) = \cosh x \Rightarrow f^{(2n-1)}(0) = 1.$$

We can conclude that only the odd derivatives will contribute to the Maclaurin series expansion, each with a denominator of $(2n-1)!$ .

The first non-zero term occurs when $n = 1$ with this choice of superscript and so we can conclude that the $n$th non-zero term is $\dfrac{x^{2n-1}}{(2n-1)!}$.

**14 a** We differentiate and evaluate at 0 until we have 2 non-zero terms.

$f(x) = \tanh x \Rightarrow f(0) = 0$

$f'(x) = \text{sech}^2 x \Rightarrow f'(0) = 1$

$f''(x) = -2\tanh x \,\text{sech}^2 x \Rightarrow f''(0) = 0$

$f'''(x) = 4\tanh^2 x \,\text{sech}^2 x - 2\,\text{sech}^4 x \Rightarrow f'''(0) = -2.$

Now we use the standard Maclaurin series expansion and obtain

$$\tanh x \approx x - \frac{2x^3}{3!}$$
$$= x - \frac{x^3}{3}.$$

**b** Using the approximation,

$$\tanh 0.8 \approx 0.8 - \frac{0.8^3}{3}$$
$$= 0.629333 (6 \text{ d.p.})$$
$$error = \left|\frac{0.629333 - \tanh 0.8}{\tanh 0.8}\right| \times 100$$
$$= 5.23\% (3 \text{ s.f.})$$

**15 a** We differentiate and evaluate at 0 until we have three non-zero terms.

$f(x) = \text{ar}\tanh x \Rightarrow f(0) = 0$

$f'(x) = \frac{1}{1-x^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{2x}{\left(1-x^2\right)^2} \Rightarrow f''(0) = 0$

$f'''(x) = \frac{2(3x^2+1)}{\left(1-x^2\right)^3} \Rightarrow f'''(0) = 2$

$f^{(4)}(x) = \frac{24x(x^2+1)}{\left(1-x^2\right)^4} \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = \frac{24(5x^4+10x^2+1)}{\left(1-x^2\right)^5} \Rightarrow f^{(5)}(0) = 24.$

Now we use the standard Maclaurin series expansion and obtain

$$\text{artanh}\, x \approx x + \frac{2x^3}{3!} + \frac{24x^5}{5!}$$
$$= x + \frac{x^3}{3} + \frac{x^5}{5}.$$

**b** By observation, the $n$ th non-zero term in the series expansion appears to be $\frac{x^{2n-1}}{2n-1}$.

**15 c** We differentiate and evaluate at 0 until we have two non-zero terms.

$f(x) = \cosh x \operatorname{artanh} x \Rightarrow f(0) = 0$

$f'(x) = \dfrac{\cosh x}{1-x^2} + \sinh x \operatorname{artanh} x \Rightarrow f'(0) = 1$

$f''(x) = \dfrac{2\sinh x}{1-x^2} + \dfrac{2x\cosh x}{(1-x^2)^2} + \cosh x \operatorname{artanh} x \Rightarrow f''(0) = 0$

$f'''(x) = \dfrac{6x\sinh x + 2\cosh x}{(1-x^2)^2} + \dfrac{8x^2\cosh x}{(1-x^2)^3}$

$+ \dfrac{3\cosh x}{1-x^2} + \sinh x \operatorname{artanh} x \Rightarrow f'''(0) = 5.$

Now we use the standard Maclaurin series expansion and obtain

$$\cosh x \operatorname{artanh} x \approx x + \frac{5x^3}{3!}$$
$$= x + \frac{5x^3}{6}$$

A less tedious way of doing this would be to take the expansions of $\cosh x$ and $\operatorname{artanh} x$ then multiply together, omitting higher order terms.

$$\cosh x \approx 1 + \frac{x^2}{2} + \frac{x^4}{24}$$
$$\operatorname{artanh} x \approx x + \frac{x^3}{3} + \frac{x^5}{5}$$
$$\cosh x \operatorname{artanh} x \approx x + \frac{x^3}{2} + \frac{x^3}{3} + \ldots$$
$$= x + \frac{5x^3}{6}.$$

**16** We differentiate and evaluate at 0 until we have three non-zero terms.

$f(x) = \sinh x \cosh 2x \Rightarrow f(0) = 0$

$f'(x) = \cosh x \cosh 2x + 2\sinh x \sinh 2x \Rightarrow f'(0) = 1$

$f''(x) = 5\sinh x \cosh 2x + 4\cosh x \sinh 2x \Rightarrow f''(0) = 0$

$f'''(x) = 13\cosh x \cosh 2x + 14\sinh x \sinh 2x \Rightarrow f'''(0) = 13$

$f^{(4)}(x) = 41\sinh x \cosh 2x + 40\cosh x \sinh 2x \Rightarrow f^{(4)}(0) = 0$

$f^{(5)}(x) = 121\cosh x \cosh 2x + 122\sinh x \sinh 2x \Rightarrow f^{(5)}(0) = 121.$

Now we use the standard Maclaurin series expansion and obtain

$$\sinh x \cosh 2x \approx x + \frac{13x^3}{3!} + \frac{121x^5}{5!}$$
$$= x + \frac{13x^3}{6} + \frac{121x^5}{120}.$$

**17 a** We differentiate $y = \cos x \cosh x$ with respect to $x$ four times.

$y = \cos x \cosh x$

$\frac{dy}{dx} = \cos x \sinh x - \sin x \cosh x$

$\frac{d^2y}{dx^2} = -2\sin x \sinh x$

$\frac{d^3y}{dx^3} = -2(\cos x \sinh x + \sin x \cosh x)$

$\frac{d^4y}{dx^4} = -4\cos x \cosh x = -4y.$

**b** We evaluate the differentials at 0 until we have three non-zero terms.

$f(x) = \cos x \cosh x \Rightarrow f(0) = 1$

$f'(x) = \cos x \sinh x - \sin x \cosh x \Rightarrow f'(0) = 0$

$f''(x) = -2\sin x \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = -2(\cos x \sinh x + \sin x \cosh x) \Rightarrow f'''(0) = 0$

$f^{(4)}(x) = -4\cos x \cosh x = -4f(x) \Rightarrow f^{(4)}(0) = -4.$

Since we have the relation $\frac{d^4y}{dx^4} = -4y$, we can conclude that all differentials that are not of the form $\frac{d^{4n}y}{dx^{4n}}$, (where $n$ is an integer) is 0 when evaluated at 0. So our third non-zero term is

$$\begin{aligned}\frac{d^8y}{dx^8} &= \frac{d^4}{dx^4}\left(\frac{d^4y}{dx^4}\right) \\ &= \frac{d^4}{dx^4}(-4y) \\ &= -4\frac{d^4y}{dx^4} \\ &= 16y \\ &= 16\cos x \cosh x \Rightarrow f^{(8)}(0) = 16.\end{aligned}$$

Now we use the standard Maclaurin series expansion and obtain

$$\begin{aligned}\cos x \cosh x &\approx 1 - \frac{4x^4}{4!} + \frac{16x^8}{8!} \\ &= 1 - \frac{x^4}{6} + \frac{x^8}{2520}\end{aligned}$$

**17 c** From the previous question, we know that the $\frac{d^{4n}y}{dx^{4n}}$ terms are the only non-zero contributions.

Recalling that the Maclaurin series expansion is $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)x^k}{k!}$ and considering

$$\begin{aligned}
\frac{d^{4n}y}{dx^{4n}} &= \frac{d^{4n-4}}{dx^{4n-4}}\left(\frac{d^4y}{dx^4}\right)\\
&= \frac{d^{4n-4}}{dx^{4n-4}}(-4y)\\
&= -4\frac{d^{4n-8}}{dx^{4n-8}}\left(\frac{d^4y}{dx^4}\right)\\
&= -4\frac{d^{4n-8}}{dx^{4n-8}}(-4y)\\
&\vdots\\
&= (-4)^n y,
\end{aligned}$$

along with all differentials that are not of the form $\frac{d^{4n}y}{dx^{4n}}$ (where $n$ is an integer) is 0 when evaluated at 0, we can write an expression.

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)x^k}{k!}$$

$$\begin{aligned}
\cos x\cosh x &= \sum_{r=0}^{\infty}\frac{\frac{d^{4r}y}{dx^{4r}}(0)x^{4r}}{(4r)!}\\
&= \sum_{r=0}^{\infty}\frac{(-4)^r y(0)x^{4r}}{(4r)!}\\
&= \sum_{r=0}^{\infty}\frac{(-4)^r x^{4r}}{(4r)!}
\end{aligned}$$

**Challenge**

We differentiate and evaluate at 0 until we have three non-zero terms.

$f(x)=\operatorname{sech} x \Rightarrow f(0)=1$

$f'(x)=-\tanh x\operatorname{sech} x \Rightarrow f'(0)=0$

$f''(x)=\tanh^2 x\operatorname{sech} x-\operatorname{sech}^3 x \Rightarrow f''(0)=-1$

$$\begin{aligned}
f'''(x) &= 2\tanh x\operatorname{sech}^3 x-\tanh^3 x\operatorname{sech} x+3\tanh x\operatorname{sech}^3 x\\
&= 5\tanh x\operatorname{sech}^3 x-\tanh^3 x\operatorname{sech} x \Rightarrow f'''(0)=0
\end{aligned}$$

$$\begin{aligned}
f^{(4)}(x) &= 5\operatorname{sech}^5 x-15\tanh^2 x\operatorname{sech}^3 x\\
&\quad -3\tanh^2 x\operatorname{sech}^3 x+\tanh^4 x\operatorname{sech} x \Rightarrow f^{(4)}(0)=5.
\end{aligned}$$

Now we use the standard Maclaurin series expansion and obtain

$$\begin{aligned}
\sinh x\cosh 2x &\approx 1-\frac{x^2}{2!}+\frac{5x^4}{4!}\\
&= 1-\frac{x^2}{2}+\frac{5x^4}{24}
\end{aligned}$$