## Hyperbolic Functions 6C

**1 a** $\text{RHS} = 2\sinh A\cosh A$

$$= 2\left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{A}+e^{-A}}{2}\right)$$

$$= \frac{1}{2}\left(e^{2A}-1+1-e^{-2A}\right)$$

$$= \frac{e^{2A}-e^{-2A}}{2}$$

$$= \sinh 2A = \text{LHS}$$

**b** $\text{RHS} = \cosh A\cosh B - \sinh A\sinh B$

$$= \left(\frac{e^{A}+e^{-A}}{2}\right)\left(\frac{e^{B}+e^{-B}}{2}\right) - \left(\frac{e^{A}-e^{-A}}{2}\right)\left(\frac{e^{B}-e^{-B}}{2}\right)$$

$$= \frac{e^{A+B}+e^{-A+B}+e^{A-B}+e^{-A-B}}{4}$$

$$- \frac{e^{A+B}-e^{-A+B}-e^{A-B}+e^{-A-B}}{4}$$

$$= \frac{2\left(e^{-A+B}+e^{A-B}\right)}{4}$$

$$= \frac{e^{A-B}+e^{-(A-B)}}{2}$$

$$= \cosh(A-B) = \text{LHS}$$

**c** $\text{RHS} = 4\cosh^3 A - 3\cosh A$

$$= 4\left(\frac{e^{A}+e^{-A}}{2}\right)^3 - 3\left(\frac{e^{A}+e^{-A}}{2}\right)$$

$$\left(e^{A}+e^{-A}\right)^3 = e^{3A}+3e^{2A}e^{-A}+3e^{A}e^{-2A}+e^{-3A}$$

$$= e^{3A}+3e^{A}+3e^{-A}+e^{-3A}$$

Use the expansion $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$.

$$\text{RHS} = \frac{e^{3A}+3e^{A}+3e^{-A}+e^{-3A}}{2} - \frac{3(e^{A}+e^{-A})}{2}$$

$$= \frac{e^{3A}+e^{-3A}}{2}$$

$$= \cosh 3A = \text{LHS}$$

**1 d** $\text{RHS} = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$

$= 2\left(\frac{e^{\frac{A-B}{2}} - e^{\frac{-A+B}{2}}}{2}\right)\left(\frac{e^{\frac{A+B}{2}} + e^{\frac{-A-B}{2}}}{2}\right)$

$= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} - e^{\frac{-A+B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}+\frac{A-B}{2}} - e^{\frac{-A+B}{2}+\frac{-A-B}{2}}\right)$

$= \frac{1}{2}\left(e^{A} - e^{B} + e^{-B} - e^{-A}\right)$

$= \frac{1}{2}\left(e^{A} - e^{-A}\right) - \frac{1}{2}\left(e^{B} - e^{-B}\right)$

$= \sinh A - \sinh B$

$= \text{LHS}$

**2 a** $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$\sinh(A-B) = \sinh A\cosh B - \cosh A\sinh B$

Replace sin *x* by sinh *x* and cos *x* by cosh *x*.

**b** $\sin 3A = 3\sin A - 4\sin^3 A$

$= 3\sin A - 4\sin A\sin^2 A$

$\sinh 3A = 3\sinh A + 4\sinh^3 A$

Replace $\sin^2 A$, the product of two sine terms, by $-\sinh^2 A$.

**c** $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\cosh A + \cosh B = 2\cosh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right)$

Replace cos *x* by cosh *x*.

**d** $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

$\cosh 2A = \frac{1+\tanh^2 A}{1-\tanh^2 A}$

$\tan^2 A = \frac{\sin^2 A}{\cos^2 A}$, so there is a product of two sines.

Replace $\tan^2 A$ by $-\tanh^2 A$.

**e** $\cos 2A = \cos^4 A - \sin^4 A$

$= \cos^4 A - (\sin^2 A)(\sin^2 A)$

$\cosh 2A = \cosh^4 A - (-\sinh^2 A)(-\sinh^2 A)$

$= \cosh^4 A - \sinh^4 A$

Replace $\sin^2 A$ by $-\sinh^2 A$.

**3 a** Using $\cosh^2 x - \sinh^2 x = 1$

$4 - \sinh^2 x = 1$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

Both positive and negative values of sinh $x$ are possible.

**b** Using $\tanh x = \dfrac{\sinh x}{\cosh x}$

$\tanh x = \pm\dfrac{\sqrt{3}}{2}$

**c** Using $\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = (2\times 4) - 1$

$= 7$

**4 a** Using $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - (-1)^2 = 1$

$\cosh^2 x = 2$

$\cosh x = \sqrt{2}$

cosh $x$ cannot be negative.

**b** Using $\sin 2x = 2\sinh x\cosh x$

$\sinh 2x = 2\times(-1)\times\sqrt{2}$

$= -2\sqrt{2}$

**c** Using $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

Alternatively use $\dfrac{\sinh 2x}{\cosh 2x} = \dfrac{\sinh 2x}{2\cosh^2 x - 1}$

$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{-1}{\sqrt{2}}$

$\tanh 2x = \dfrac{\left(-\frac{2}{\sqrt{2}}\right)}{1+\left(\frac{1}{2}\right)}$

$= \dfrac{-2}{\sqrt{2}}\times\dfrac{2}{3}$

$= \dfrac{-4}{3\sqrt{2}} = -\dfrac{2\sqrt{2}}{3}$

**5 a** $3\sinh x + 4\cosh x = 4$

$\frac{3(e^x - e^{-x})}{2} + \frac{4(e^x + e^{-x})}{2} = 4$

$3e^x - 3e^{-x} + 4e^x + 4e^{-x} = 8$

$7e^x - 8 + e^{-x} = 0$ ← Multiply throughout by $e^x$.

$7e^{2x} - 8e^x + 1 = 0$

$(7e^x - 1)(e^x - 1) = 0$ ← Solve as a quadratic in $e^x$.

$e^x = \frac{1}{7}$ or $e^x = 1$

$x = \ln\left(\frac{1}{7}\right),\ x = 0$ ← Note that $\ln\left(\frac{1}{7}\right) = \ln(7^{-1}) = -\ln 7$

**b** $7\sinh x - 5\cosh x = 1$

$\frac{7(e^x - e^{-x})}{2} - \frac{5(e^x + e^{-x})}{2} = 1$

$7e^x - 7e^{-x} - 5e^x - 5e^{-x} = 2$

$2e^x - 2 - 12e^{-x} = 0$ ← Multiply throughout by $e^x$.

$e^x - 1 - 6e^{-x} = 0$

$e^{2x} - e^x - 6 = 0$

$(e^x - 3)(e^x + 2) = 0$

$e^x = 3$ ← $e^x = -2$ is not possible for real $x$.

$x = \ln 3$

**c** $30\cosh x = 15 + 26\sinh x$

$30\frac{(e^x + e^{-x})}{2} = 15 + 26\frac{(e^x - e^{-x})}{2}$

$15e^x + 15e^{-x} = 15 + 13e^x - 13e^{-x}$

$2e^x - 15 + 28e^{-x} = 0$ ← Multiply throughout by $e^x$.

$2e^{2x} - 15e^x + 28 = 0$

$(2e^x - 7)(e^x - 4) = 0$ ← Solve as a quadratic in $e^x$.

$e^x = \frac{7}{2},\ e^x = 4$

$x = \ln\left(\frac{7}{2}\right),\ x = \ln 4$

**5 d**

$$13\sinh x - 7\cosh x + 1 = 0$$
$$13\frac{(e^x - e^{-x})}{2} - 7\frac{(e^x + e^{-x})}{2} + 1 = 0$$
$$13e^x - 13e^{-x} - 7e^x - 7e^{-x} + 2 = 0$$
$$6e^x + 2 - 20e^{-x} = 0$$
$$3e^x + 1 - 10e^{-x} = 0$$

Multiply throughout by $e^x$.

$$3e^{2x} + e^x - 10 = 0$$
$$(3e^x - 5)(e^x + 2) = 0$$

Solve as a quadratic in $e^x$.

$$e^x = \frac{5}{3}$$

$e^x = -2$ is not possible for real $x$.

$$x = \ln\left(\frac{5}{3}\right)$$

**e** $\cosh 2x - 5\sinh x = 13$

Using $\cosh 2x = 1 + 2\sinh^2 x$,

$$1 + 2\sinh^2 x - 5\sinh x = 13$$

Use $\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1})$.

$$2\sinh^2 x - 5\sinh x - 12 = 0$$
$$(2\sinh x + 3)(\sinh x - 4) = 0$$
$$\sinh x = -\frac{3}{2},\ \sinh x = 4$$
$$x = \operatorname{arsinh}\left(-\frac{3}{2}\right),\ x = \operatorname{arsinh} 4$$
$$x = \ln\left(-\frac{3}{2} + \sqrt{\frac{9}{4} + 1}\right)$$
$$= \ln\left(\frac{-3 + \sqrt{13}}{2}\right)$$
$$x = \ln(4 + \sqrt{16 + 1})$$
$$= \ln(4 + \sqrt{17})$$

**5 f** $3\sinh^2 x - 13\cosh x + 7 = 0$

Using $\cosh^2 x - \sinh^2 x = 1$,

$3(\cosh^2 x - 1) - 13\cosh x + 7) = 0$

$3\cosh^2 x - 13\cosh x + 4 = 0$

$(3\cosh - 1)(\cosh x - 4) = 0$

$\cosh x = \frac{1}{3},\ \cosh x = 4$

$\cosh x \geq 1$, so $\cosh x = \frac{1}{3}$ is not possible.

$\cosh x = 4$

$x = \text{arcosh}\,4,\ -\text{arcosh}\,4$

Use $\text{arcosh}\,x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln\,(x - \sqrt{x^2 - 1})$ is also a solution.

$x = \ln(4 \pm \sqrt{4^2 - 1})$

$= \ln(4 \pm \sqrt{15})$

**g** $\sinh 2x - 7\sinh x = 0$

$2\sinh x\cosh x - 7\sin x = 0$

$\sinh x(2\cosh x - 7) = 0$

$\sinh x = 0,\ \cosh x = \frac{7}{2}$

$x = 0,\ x = \pm\,\text{arcosh}\left(\frac{7}{2}\right)$

Use $\text{arcosh}\,x = \ln(x + \sqrt{x^2 - 1})$, but remember that $\ln(x - \sqrt{x^2 - 1})$ is also a solution.

$\text{arcosh}\left(\frac{7}{2}\right) = \ln\left(\frac{7}{2} + \sqrt{\frac{49}{4} - 1}\right)$

$= \ln\left(\frac{7 + \sqrt{45}}{2}\right)$

$= \ln\left(\frac{7 + 3\sqrt{5}}{2}\right)$

$x = 0,\ x = \ln\left(\frac{7 \pm 3\sqrt{5}}{2}\right)$

**h** $4\cosh x + 13e^{-x} = 11$

$4\frac{(e^x + e^{-x})}{2} + 13e^{-x} = 11$

$2e^x + 2e^{-x} + 13e^{-x} = 11$

Multiply throughout by $e^x$.

$2e^x + 15e^{-x} - 11 = 0$

$2e^{2x} - 11e^x + 15 = 0$

$(2e^x - 5)(e^x - 3) = 0$

Solve as a quadratic in $e^x$.

$e^x = \frac{5}{2},\ e^x = 3$

$x = \ln\left(\frac{5}{3}\right),\ x = \ln 3$

**5 i**

$$2\tanh x = \cosh x$$

$$\frac{2\sinh}{\cosh x} = \cosh x$$

$$2\sinh x = \cosh^2 x$$

Using $\cosh^2 x - \sinh^2 x = 1$

$$2\sinh x = 1 + \sinh^2 x$$

$$\sinh^2 x - 2\sinh x + 1 = 0$$

$$(\sinh x - 1)^2 = 0$$

$$\sinh x = 1$$

$$x = \text{arsinh}\,1$$

Use $\text{arsinh}\,x = \ln(x + \sqrt{x^2+1})$.

$$x = \ln(1 + \sqrt{1^2+1})$$

$$= \ln(1 + \sqrt{2})$$

**6 a**

$$2\cosh^2 x - 1 = 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= 2\left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x.$$

**b** Using the relation $\cosh 2x = 2\cosh^2 x - 1$, the equation becomes $2\cosh^2 x - 3\cosh x - 9 = 0$.
Substituting $u = \cosh x$ into this equation allows us to clearly see a quadratic equation

$2u^2 - 3u - 9 = 0$ which gives solutions $u = \frac{3 \pm \sqrt{9+72}}{4}$, we neglect the negative solution to avoid complex numbers.

i.e.

$u = 3$.

This means that our solutions for $x$ are the values for

$x_1 = \text{ar}\cosh(3)$.

Now recalling the expression $\text{ar}\cosh u = \ln\left(u \pm \sqrt{u^2 - 1}\right)$, we obtain the exact logarithm solutions

$x = \ln\left(3 \pm 2\sqrt{2}\right)$.

**7** Using the identity $\cosh^2 x - \sinh^2 x \equiv 1$, the equation in question becomes
$2(\cosh^2 x - 1) - 5\cosh x = 5$.
We simplify and use the substitution $u = \cosh x$ in order to obtain $2u^2 - 5u - 7 = 0$. This equation has solutions $u = \frac{5 \pm \sqrt{25+56}}{4}$, we neglect the negative solution to avoid complex numbers, thus $u = \frac{7}{2}$.
This means that our solutions for $x$ are the values for
$x = \text{ar}\cosh\left(\frac{7}{2}\right)$.
Now recalling the expression $\text{ar}\cosh u = \ln\left(u \pm \sqrt{u^2 - 1}\right)$, we obtain the exact logarithm solutions
$x = \ln\left(\frac{1}{2}\left(7 \pm 3\sqrt{5}\right)\right)$.

**8** $\text{sech}^2 x \equiv 1 + \tanh^2 x$ is not true, the correct identity is $\text{sech}^2 x \equiv 1 - \tanh^2 x$ by Osborn's rule.
He has split the fraction via denominator on the second line. This is not valid.
In mathematics we write this as
$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.
The final mistake he made was assuming that taking the reciprocal of all terms preserves the value.
This is incorrect. For example $3 + 1 \neq \frac{1}{3} + 1$.
The correct proof is

$$\begin{aligned}\frac{1+\tanh^2 x}{1-\tanh^2 x} &\equiv \frac{2-\text{sech}^2 x}{\text{sech}^2 x} \\ &\equiv \frac{2}{\text{sech}^2 x} - 1 \\ &\equiv 2\cosh^2 x - 1.\end{aligned}$$

**9 a** Recall the identity $R\cosh(x+a) \equiv R\cosh x \cosh a + R\sinh x \sinh a$.
From this we can set $R\cosh a = 10$ and $R\sinh a = 6$.
In order to find values for $R$ and $a$:
$\frac{R\sinh a}{R\cosh a} = \tanh a = \frac{6}{10} = \frac{3}{5}$
giving

$$\begin{aligned}a &= \text{artanh}(0.6) \\ &\approx 0.693 (3 \text{ d.p.})\end{aligned}$$

We can now use the expression $R\cosh a = 10$.
To find $\cosh a$ : $\text{sech}^2 a = 1 - \tanh^2 a = 1 - \left(\frac{3}{5}\right)^2 = \frac{16}{25}$
$\therefore \cosh^2 a = \frac{1}{\text{sech}^2 a} = \frac{25}{16}$ and so $\cosh a = \frac{5}{4}$
So $R = \frac{10}{\cosh a} = \frac{10}{\left(\frac{5}{4}\right)} = 8$

Thus giving us $10\cosh x + 6\sinh x = 8\cosh(x + 0.693)$.

**9 b** The minimum value occurs when $\cosh(x+0.693)$ is minimal. We know this occurs when $x+0.693=0$, by the graph of $\cosh x$. So the minimal value is $8\cosh 0 = 8$ .

We could also find this answer by setting $y = 8\cosh(x+0.693)$, finding the solution to $\frac{dy}{dx} = 8\sinh(x+0.693) = 0$ as $x = -0.693$ and substituting in, to get $y = 8\cosh 0 = 8$.

The second derivative should be checked to be positive at this point in order to conclude that the stationary point is minimal, not maximal.

**c** $10\cosh x + 6\sinh x = 8\cosh(x+0.693) = 11$.

We rearrange the equation to be $\cosh(x+0.693) = \frac{11}{8}$.

Set $u = x + 0.693$ and now noting the symmetry about $u = 0$ we obtain two solutions,

$u = \pm\text{arcosh}\left(\frac{11}{8}\right) \approx \pm 0.841(3\text{ d.p.})$

This means our solutions for $x$ are

$x_1 = -1.534,$

$x_2 = 0.148.$