Hyperbolic Functions 6A

1 a $\sinh 4 = 27.29 \quad (2 \text{ d.p.})$

$$\left(\frac{e^{4} - e^{-4}}{2} = 27.29\right)$$

Direct from calculator.

b $\cosh\left(\frac{1}{2}\right) = 1.13 \ (2\,\text{d.p.})$

$$\left(\frac{e^{0.5} + e^{-0.5}}{2} = 1.13\right)$$

Direct from calculator.

c $\tanh(-2) = -0.96 \ (2\,\text{d.p.})$

$$\left(\frac{e^{-4} - 1}{e^{-4} + 1} = -0.96\right)$$

Direct from calculator.

2 a $\sinh 1 = \frac{e^{1} - e^{-1}}{2} = \frac{e - e^{-1}}{2}$

b $\cosh 4 = \frac{e^{4} + e^{-4}}{2}$

c $\tanh 0.5 = \frac{e^{1} - 1}{e^{1} + 1}$

$= \frac{e - 1}{e + 1}$

Use $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$.

3 a $\sinh(\ln 2) = \frac{e^{h2} - e^{-h2}}{2}$

$= \frac{2 - \frac{1}{2}}{2} = \frac{3}{4}$

$e^{h2} = 2$, and $e^{-\ln 2} = e^{\ln 2^{-1}} = \frac{1}{2}$

b $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2}$

$= \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$

$e^{\ln 3} = 3$, and $e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$

c $\tanh(\ln 2) = \frac{e^{2\ln 2} - 1}{e^{2\ln 2} + 1}$

$= \frac{4 - 1}{4 + 1} = \frac{3}{5}$

$e^{2\ln 2} = e^{\ln 2^{2}} = 4$

4 $\dfrac{e^x + e^{-x}}{2} = 2$

$e^x + e^{-x} = 4$ ← Multiply throughout by e^x.

$e^{2x} + 1 = 4e^x$

$e^{2x} - 4e^x + 1 = 0$

$e^x = \dfrac{4 \pm \sqrt{16-4}}{2}$ ← Solve as a quadratic in e^x.

$e^x = 3.732$ or $e^x = 0.268$

$x = \ln 3.732 = 1.32$ (2 d.p.)

$x = \ln 0.268 = -1.32$ (2 d.p.)

5 $\dfrac{e^x - e^{-x}}{2} = 1$

$e^x - e^{-x} = 2$ ← Multiply throughout by e^x.

$e^{2x} - 1 = 2e^x$

$e^{2x} - 2e^x - 1 = 0$

$e^x = \dfrac{2 \pm \sqrt{4+4}}{2}$ ← Solve as a quadratic in e^x.

$e^x = 2.414$ or $e^x = -0.414$

$e^x = 2.414$

$x = \ln 2.414 = 0.88$ (2 d.p.) ← e^x cannot be negative.

6 $\dfrac{e^{2x} - 1}{e^{2x} + 1} = -\dfrac{1}{2}$

$2\left(e^{2x} - 1\right) = -\left(e^{2x} + 1\right)$

$2e^{2x} - 2 = -e^{2x} - 1$

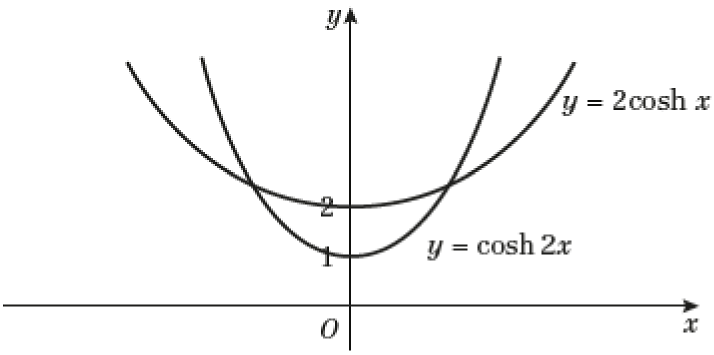
$3e^{2x} = 1$

$e^{2x} = \dfrac{1}{3}$

$2x = \ln\left(\dfrac{1}{3}\right)$

$x = \dfrac{1}{2}\ln\left(\dfrac{1}{3}\right) = -0.55$ (2 d.p.)

7



For $f(x) = \cosh x$,
$f(2x) = \cosh 2x$, a horizontal sketch of scale factor $\frac{1}{2}$.

For $f(x) = \cosh x$,
$2f(x) = 2\cosh x$, a vertical stretch of scale factor 2.

8 **a** $f(x) \in \mathbb{R}$ (All real numbers)

b $f(x) \geqslant 1$

c $-1 < f(x) < 1$

$|f(x)| < 1$

Check the graph of each hyperbolic function to see which y values are possible.

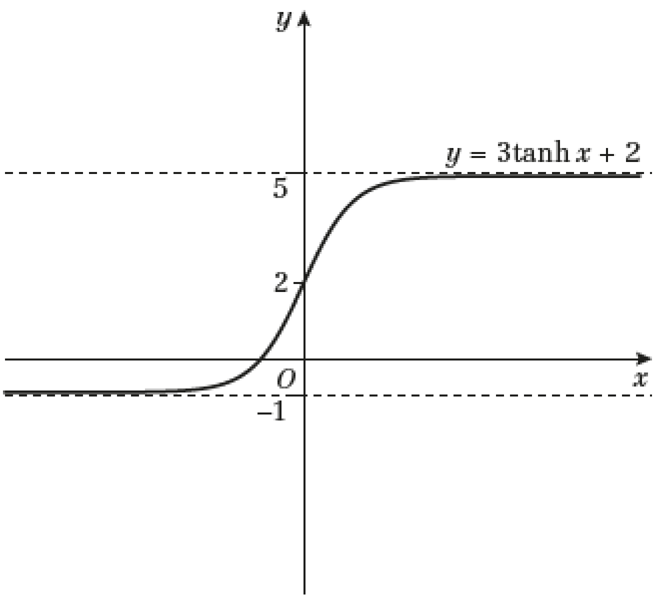
9 **a** $y = 3\tanh x + 2 = 3\dfrac{\sinh x}{\cosh x} + 2$

When $x = 0$, $y = 3\dfrac{\sinh 0}{\cosh 0} + 2 = 2$.

When x is large and positive, $\sinh x \approx \frac{1}{2}e^x$ and $\cosh x \approx \frac{1}{2}e^x$, so $y \approx 5$.

When x is large and negative, $\sinh x \approx -\frac{1}{2}e^{-x}$ and $\cosh x \approx \frac{1}{2}e^{-x}$, so $y \approx -1$.

Alternatively, the graph of $y = 3\tanh x + 2$ can be envisioned by scaling $y = \tanh x$ by a factor of 3 in the y direction, followed by a transition of 2 units in the positive y direction.



9 b When x is large and positive, $\sinh x \approx \frac{1}{2}e^x$ and $\cosh \approx \frac{1}{2}e^x$.

So $y \approx 3+2$, thus giving $y=5$ as an asymptote.

When x is large and negative, $\sinh x \approx -\frac{1}{2}e^{-x}$ and $\cosh \approx \frac{1}{2}e^{-x}$.

So $y \approx -3+2$, thus giving $y=-1$ as an asymptote.

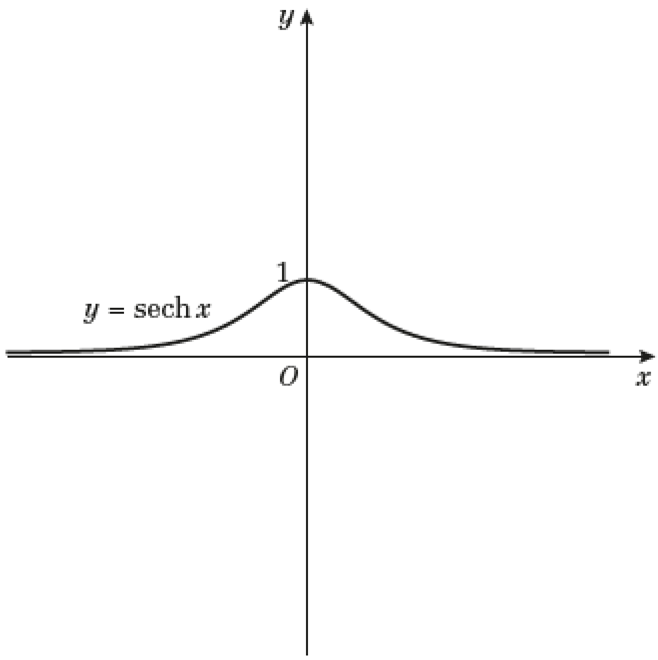
Challenge

a $y = \text{sech}\, x = \frac{1}{\cosh x}$

When $x=0$, $y = \frac{1}{\cosh 0} = 1$

When x is large and positive, $\cosh x \approx \frac{1}{2}e^x$, so $y \approx 0$.

When x is large and negative, $\cosh x \approx \frac{1}{2}e^{-x}$, so $y \approx 0$.



Challenge

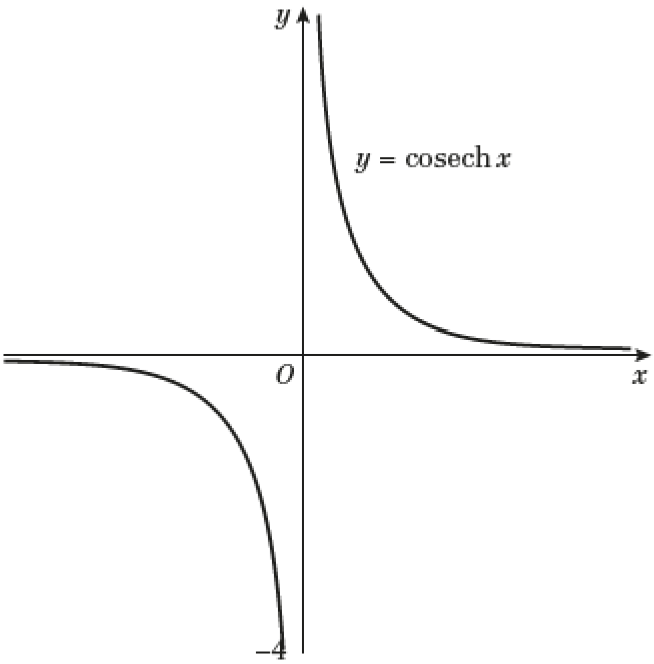
b $y = \text{cosech}\, x = \dfrac{1}{\sinh x}$ When $x = 0$, $\sinh x = 0$. So $\text{cosech}\, 0$ is not well defined.

As $x \to 0_+$ (zero from above), $\sinh x$ will be an infinitesimal positive number and so $\text{cosech}\, x \to +\infty$.

As $x \to 0_-$ (zero from below), $\sinh x$ will be an infinitesimal negative number and so $\text{cosech}\, x \to -\infty$.

When x is large and positive, $\sinh x \approx \dfrac{1}{2}e^x$, so $y \approx 0$.

When x is large and negative, $\sinh x \approx -\dfrac{1}{2}e^{-x}$, so $y \approx 0$.



Challenge

c $y = \coth x = \dfrac{\cosh x}{\sinh x}$ When $x = 0$, $\sinh x = 0$, and $\cosh x = 1$. So $\coth 0$ is not well defined.

As $x \to 0_+$ (zero from above), $\sinh x$ will be an infinitesimal positive number and $\cosh x \approx 1$, so $\operatorname{cosech} x \to +\infty$.

As $x \to 0_-$ (zero from below), $\sinh x$ will be an infinitesimal negative number and $\cosh x \approx 1$, so $\operatorname{cosech} x \to -\infty$.

When x is large and positive, $\sinh x \approx \dfrac{1}{2}e^x$ and $\cosh x \approx \dfrac{1}{2}e^x$, so $y \approx 1$.

When x is large and negative, $\sinh x \approx -\dfrac{1}{2}e^{-x}$ and $\cosh x \approx \dfrac{1}{2}e^{-x}$, so $y \approx -1$.