Review exercise 1

1 Using Euler's solution $e^{i\theta} = \cos\theta + i\sin\theta$,

$$\cos 2x + i\sin 2x = e^{i2x}$$

$$\cos 9x - i\sin 9x = \cos(-9x) + i\sin(-9x) = e^{i(-9x)}$$

Hence

$$\frac{\cos 2x + i\sin 2x}{\cos 9x - i\sin 9x} = \frac{e^{i2x}}{e^{i(-9x)}} = e^{i(2x+9x)} = e^{i11x}$$

$$= \cos 11x + i\sin 11x$$

This is the required form with $n = 11$.

For any angle, $\theta, \cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$

You will find these relations useful when finding the arguments of complex numbers.

Manipulating the arguments in $e^{i\theta}$ you use the ordinary laws of indices.

2 a By de Moivre's theorem

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5, \text{ say}$$

$$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^3 s^3 + 5ci^4 s^4 + i^5 s^5$$

$$= c^5 + i5c^4 s - 10c^3 s^3 - i10c^2 s^3 + 5cs^4 - is^5$$

It is sensible to abbreviate $\cos\theta$ and $\sin\theta$ as c and s respectively when you have as many powers of $\cos\theta$ and $\sin\theta$ to write out as you have in this question.

Use the binomial expansion.

Use $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and $i^5 = i \times i^4 = i \times 1 = i$.

Equating real parts

$$\cos 5\theta = c^5 - 10c^3 s^2 + 5cs^4$$

Using $\cos^2\theta + \sin^2\theta = 1$

$$\cos 5\theta = c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2$$

$$= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5$$

$$= 16c^5 - 20c^3 + 5c$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta, \text{ as required}$$

b Substitute $x = \cos\theta$ into $16x^5 - 20x^3 + 5x + 1 = 0$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 1 = 0$$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = -1$$

Using the result of part **a**

$$\cos 5\theta = -1$$

$$5\theta = \pi, 3\pi, 5\pi$$

$$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}$$

$$x = \cos\theta = \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi$$

$$= 0.809, -0.309, -1$$

Additional solutions are found by increasing the angles in steps of 2π. You are asked for 3 answers, so you need 3 angles at this stage.

The two approximate answers are given to 3 decimal places, as the question specified; the remaining answer –1 is exact.

3 a By de Moivre's theorem

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5 = (c + is)^5, \text{ say}$$

$$= c^5 + 5c^4 is + 10c^3 i^2 s^2 + 10c^2 i^2 s^3 + 5ci^3 s^4 + i^5 s^5$$

$$= c^5 + i5c^4 s - 10c^3 s^2 - i10c^2 s^3 + 5cs^4 - is^5$$

3 a Equating imaginary parts

$$\begin{aligned}\sin 5\theta &= 5c^4s - 10c^2s^3 + s^5\\ &= s(5c^4 - 10c^2s^2 + s^4)\\ &= s(5c^4 - 10c^2(1-c^2) + (1-c^2)^2)\\ &= s(5c^4 - 10c^2 + 10c^4 + 1 - 2c^2 + c^4)\\ &= s(16c^4 - 12c^2 + 1)\\ &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1), \text{ as required}\end{aligned}$$

Repeatedly using the identity $\cos^2\theta + \sin^2\theta = 1$, which in this context is $s^2 = 1 - c^2$.

b

$$\sin 5\theta + \cos\theta \sin 2\theta = 0$$
$$\sin\theta(16\cos^4\theta - 12\cos^2\theta + 1) + 2\sin\theta\cos^2\theta = 0$$
$$\sin\theta(16\cos^4\theta - 10\cos^2\theta + 1) = 0$$
$$\sin\theta(2\cos^2\theta - 1)(8\cos^2\theta - 1) = 0$$

Using the identity proved in part **a** and the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Hence $\sin\theta = 0,\ \cos^2\theta = \frac{1}{2},\ \cos^2\theta = \frac{1}{8}$

$\sin\theta = 0 \Rightarrow \theta = 0$

$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}$

You must consider the negative as well as the positive square roots.

$\cos^2\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2\sqrt{2}}$

$\cos\theta = \frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.209$ (3 d.p.)

$\cos\theta = -\frac{1}{2\sqrt{2}} \Rightarrow \theta = 1.932$ (3 d.p.)

The question has specified no accuracy and any sensible accuracy would be accepted for the approximate answers.

The solutions of the equation are

$0,\ \frac{\pi}{4},\ \frac{3\pi}{4},\ 1.209$ (3 d.p.) and 1.932 (3 d.p.).

4 a $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

Let $z = e^{i\theta}$

Putting $z = e^{i\theta}$ shortens the working.

then $\sin\theta = \dfrac{z - z^{-1}}{2i}$

$$\sin^5\theta = \left(\frac{z - z^{-1}}{2i}\right)^5$$

Use Pascal's triangle to remember the coefficients in $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$= \frac{1}{(2i)^5}(z^5 - 5z^4 \times z^{-1} + 10z^3 \times z^{-2} - 10z^2 \times z^{-3} + 5z \times z^{-4} - z^{-5})$$

$$= \frac{1}{32i}(z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5})$$

$$= \frac{1}{16}\left(\frac{z^5 - z^{-5}}{2i} - \frac{5(z^3 - z^{-3})}{2i} + \frac{10(z - z^{-1})}{2i}\right)$$

The general relation is $\sin n\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i} = \dfrac{z^n - z^{-n}}{2i}$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta), \text{ as required}$$

b $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\frac{\pi}{2}} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$

Each term on the right hand side of the identity shown in part **a** can be integrated using the formula $\int \sin n\theta \, d\theta = -\dfrac{\cos n\theta}{n}$.

$$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(0 - \left(-\frac{1}{5} + \frac{5}{3} - 10\right)\right)$$

$$= \frac{1}{16} \times \frac{128}{15} = \frac{8}{15}, \text{ as required}$$

5 a $z = \cos\theta + i\sin\theta$

Using de Moivre's theorem

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ **(1)**

From **(1)**

$$z^{-n} = \frac{1}{z^n} = \frac{1}{\cos n\theta + i\sin n\theta}$$

$$= \frac{1}{\cos n\theta + i\sin n\theta} \times \frac{\cos n\theta - i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

Multiply the numerator and denominator by $\cos n\theta - i\sin n\theta$, the conjugate complex number of $\cos n\theta + i\sin n\theta$.

$$= \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta} = \cos n\theta - i\sin n\theta \quad \textbf{(2)}$$

Use $\cos^2 n\theta + \sin^2 n\theta = 1$.

$$z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta, \text{ as required.}$$

5 b $\cos\theta = \dfrac{e^{i\theta}+e^{i\theta}}{2} = \dfrac{z+z^{-1}}{2}$

$$\cos^6\theta = \left(\frac{z+z^{-1}}{2}\right)^6$$

$$= \frac{1}{64}(z^6 + 6z^5z^{-1} + 15z^4z^{-2} + 20z^3z^{-3} + 15z^2z^{-4} + 6z^1z^{-5} + z^{-6})$$

Pair the terms as shown.

$$= \frac{1}{64}(z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$$

$$= \frac{1}{32}\left(\frac{z^6+z^{-6}}{2} + \frac{6(z^4+z^{-4})}{2} + \frac{15(z^2+z^{-2})}{2} + \frac{20}{2}\right)$$

$$= \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

You use $\dfrac{z^n+z^{-n}}{2} = \cos n\theta$ with $n = 6, 4$ and 2.

c $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = \frac{1}{32}\int_0^{\frac{\pi}{2}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$

$$= \frac{1}{32} - \left[\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{32} \times 10 \times \frac{\pi}{2} = \frac{5\pi}{32}, \text{ as required}$$

With the exception of 10θ all of these terms have value 0 at both the upper and the lower limit.

6 $\displaystyle C + iS = \sum_{r=0}^{n-1} e^{ir\theta} = \frac{e^{in\theta}-1}{e^{i\theta}-1} = \frac{\cos n\theta + i\sin n\theta - 1}{\cos\theta + i\sin\theta - 1}$

$$= \frac{(\cos n\theta - 1 + i\sin n\theta)(\cos\theta - 1 - i\sin\theta)}{(\cos\theta - 1 + i\sin\theta)(\cos\theta - 1 - i\sin\theta)}$$

$$= \frac{\cos n\theta\cos\theta - \cos\theta - \cos n\theta + 1 + \sin n\theta\sin\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$+ i\frac{\sin\theta - \cos n\theta\sin\theta + \sin n\theta\cos\theta - \sin n\theta}{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta}$$

$$C = \text{Re}(C+iS) = \frac{\cos(n-1)\theta - \cos\theta - \cos n\theta + 1}{2 - 2\cos\theta}$$

$$S = \text{Im}(C+iS) = \frac{\sin(n-1)\theta + \sin\theta - \sin n\theta}{2 - 2\cos\theta}$$

7 a Let $4+4\text{i} = r(\cos\theta + \text{i}\sin\theta) = r\cos\theta + \text{i}r\sin\theta$

Equating real parts

$4 = r\cos\theta$ **(1)**

Equating imaginary parts

$4 = r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Substituting $\theta = \frac{\pi}{4}$ into **(1)**

$4 = r\cos\frac{\pi}{4} \Rightarrow 4 = r \times \frac{1}{\sqrt{2}} \Rightarrow r = 4\sqrt{2}$

Hence

$4+4\text{i} = 4\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

$= 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{\pi}{4}}$

$z^5 = 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{\pi}{4}}, 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}, 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{17\pi}{4}}, 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{25\pi}{4}}, 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{33\pi}{4}}$

$z = 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{\pi}{20}}, 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{9\pi}{20}}, 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{17\pi}{20}}, 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{25\pi}{20}}, 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{33\pi}{20}}$

This is the required form with $r = \sqrt{2}$ and

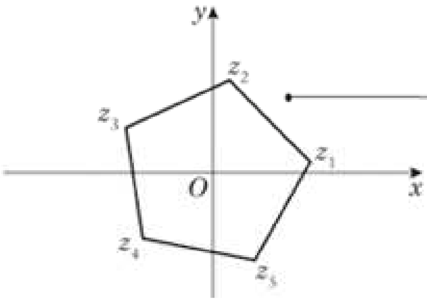
$k = \frac{1}{20}, \frac{9}{20}, \frac{17}{20}, \frac{25}{20}\left(=\frac{5}{4}\right), \frac{33}{20}.$

Finding the roots of a complex number is usually easier if you obtain the number in the form $r\text{e}^{\text{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta + \text{i}\sin\theta)$.

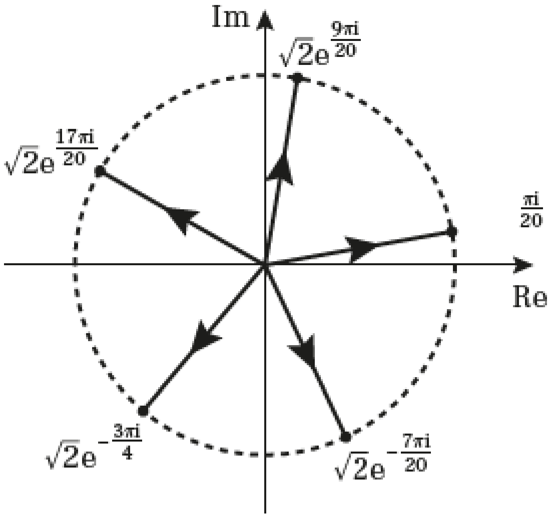
To take the fifth root, write $4\sqrt{2} = 2^{\frac{5}{2}}$.

For example, if $z^5 = 2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}$ then $z = \left(2^{\frac{5}{2}}\text{e}^{\text{i}\frac{9\pi}{4}}\right)^{\frac{1}{5}} = 2^{\frac{5}{2}\times\frac{1}{5}}\text{e}^{\text{i}\frac{9\pi}{4}\times\frac{1}{5}} = 2^{\frac{1}{2}}\text{e}^{\text{i}\frac{9\pi}{20}}$.

7 b



The points representing the 5 roots are the vertices of a regular pentagon.



8 a Let $32+32\sqrt{3}\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)=r\cos\theta+\mathrm{i}r\sin\theta$

Finding the roots of a complex number is usually easier if you obtain the number in the form $re^{\mathrm{i}\theta}$. As you will use Euler's relation, the first step towards this is to get the complex number into the form $r(\cos\theta+\mathrm{i}\sin\theta)$.

Equating real parts

$32=r\cos\theta$ **(1)**

Equating imaginary parts

$32\sqrt{3}=r\sin\theta$ **(2)**

Dividing **(2)** by **(1)**

$$\tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$$

Substituting $\theta=\frac{\pi}{3}$ into **(1)**

$$32=r\cos\frac{\pi}{3}\Rightarrow 32=r\times\frac{1}{2}\Rightarrow r=64$$

Hence

$$32+32\sqrt{3}\mathrm{i}=64\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$$

$$=64e^{\mathrm{i}\frac{\pi}{3}}$$

$$z^3=64e^{\mathrm{i}\frac{\pi}{3}},\ 64e^{\mathrm{i}\frac{7\pi}{3}},\ 64e^{\mathrm{i}\frac{-5\pi}{3}}$$

$$z=4e^{\mathrm{i}\frac{\pi}{9}},\ 4e^{\mathrm{i}\frac{7\pi}{9}},\ 4e^{-\mathrm{i}\frac{5\pi}{9}}$$

Additional solutions are found by increasing or decreasing the arguments in steps of 2π. You area asked for 3 answers, so you need 3 arguments. Had you increased the argument $\frac{7\pi}{3}$ by 2π to $\frac{13\pi}{3}$, this would have given a correct solution to the equation but it would lead to $\theta=\frac{13\pi}{9}$, which does not satisfy the condition $\theta\le\pi$ in the question. So the third argument has to be found by subtracting 2π from $\frac{\pi}{3}$.

The solutions are $re^{\mathrm{i}\theta}$ where $r=4$ and

$$\theta=-\frac{5\pi}{9},\frac{\pi}{9},\frac{7\pi}{9}$$

b $z=4e^{\mathrm{i}\frac{\pi}{9}},\ 4e^{\mathrm{i}\frac{7\pi}{9}},\ 4e^{-\mathrm{i}\frac{5\pi}{9}}$

$$z^9=\left(4e^{\mathrm{i}\frac{\pi}{9}}\right)^9,\ \left(4e^{\mathrm{i}\frac{7\pi}{9}}\right)^9,\ \left(4e^{-\mathrm{i}\frac{5\pi}{9}}\right)^9$$

$$=4^9e^{\mathrm{i}\pi},\ 4^9e^{\mathrm{i}7\pi},\ 4^9e^{-\mathrm{i}5\pi}$$

$e^{\mathrm{i}\pi}=\cos\pi+\mathrm{i}\sin\pi=-1+0=-1$. Similarly for the arguments 7π and -5π.

The value of all three of these expressions is $-4^9=-2^{18}$

Hence the solutions satisfy $z^9+2^k=0$, where $k=18$.

9 $\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$

The modulus of the complex number i is 1 and its argument is $\frac{\pi}{2}$. So $\mathrm{i} = 1\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$.

$z^5 = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{2}},$

Additional solutions are found by increasing the arguments in steps of 2π. As the equation is of degree 5, there are exactly 5 distinct answers.

$z = \mathrm{e}^{\mathrm{i}\frac{\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{13\pi}{10}}, \mathrm{e}^{\mathrm{i}\frac{17\pi}{10}}$

For example, if $z^5 = \mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}$ then $z = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{2}}\right)^{\frac{1}{5}} = \left(\mathrm{e}^{\mathrm{i}\frac{9\pi}{10}}\right)$.

Hence

$z = \cos\theta + \mathrm{i}\sin\theta$, where

$\theta = \frac{\pi}{10}, \frac{5\pi}{10}\left(=\frac{\pi}{2}\right), \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

10 a $z^5 = 16 + 16i\sqrt{3} = 32\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)$

as $\sqrt{16^2 + \left(16\sqrt{3}\right)^2} = 32, \arctan\frac{16\sqrt{3}}{16} = \frac{\pi}{3}$

$z = r(\cos\theta + i\sin\theta)$

$z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

$r^5 = 32, r = 2$

$\theta = \frac{\pi}{15} + \frac{2k\pi}{5} = \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, -\frac{\pi}{3}, -\frac{11\pi}{15}$

$z = 2e^{\frac{i\pi}{15}}, 2e^{\frac{7i\pi}{15}}, 2e^{\frac{13i\pi}{15}}, 2e^{-\frac{i\pi}{3}}, 2e^{-\frac{11i\pi}{15}}$

b The polygon is a regular pentagon.

11 a

$z^5 = 1$

$z = 1, e^{\frac{2i\pi}{5}}, e^{\frac{4i\pi}{5}}, e^{-\frac{2i\pi}{5}}, e^{-\frac{4i\pi}{5}}$

$\sum_{r=0}^{n-1} wz^r = \frac{w(z^n - 1)}{z - 1}$

$w = e^{-\frac{4i\pi}{5}}, z = e^{\frac{2i\pi}{5}}$

$1 + e^{\frac{2i\pi}{5}} + e^{\frac{4i\pi}{5}} + e^{-\frac{2i\pi}{5}} + e^{-\frac{4i\pi}{5}} = e^{-\frac{4i\pi}{5}} \frac{\left(e^{\frac{2i\pi}{5}}\right)^5 - 1}{e^{\frac{2i\pi}{5}} - 1}$

$= e^{-\frac{4i\pi}{5}} \frac{e^{2i\pi} - 1}{e^{\frac{2i\pi}{5}} - 1} = 0$

11 b $(2-3)+i(1-0)=-1+i=\sqrt{2}e^{-\frac{i\pi}{4}}$

$z=2+i+\sqrt{2}e^{-\frac{i\pi}{4}}=3$

$z=2+i+\sqrt{2}e^{-\frac{i\pi}{4}}e^{\frac{2i\pi}{5}}=2+i+\sqrt{2}e^{\frac{3i\pi}{20}}$

$=3.26+1.64i$ (2 d.p.)

$z=2+i+\sqrt{2}e^{\frac{11i\pi}{20}}=1.78+2.40i$ (2 d.p.)

$z=2+i+\sqrt{2}e^{\frac{19i\pi}{20}}=0.60+1.22i$ (2 d.p.)

$z=2+i+\sqrt{2}e^{-\frac{13i\pi}{20}}=1.36-0.26i$ (2 d.p.)

Vertices of polygon are at

$(3,0),(3.26,1.64),(1.78,2.40),$

$(0.60,1.22),(1.36,-0.26)$

12 $\frac{2}{(r+1)(r+2)}=\frac{A}{r+1}+\frac{B}{r+2}$

$2=A(r+2)+B(r+1)$

$2=A=-B$

Let $f(r)=\frac{1}{r+1}$

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+2)}=2\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)$$

$$=2(f(1)-f(n+1))$$

$$=2\left(\frac{1}{2}-\frac{1}{n+2}\right)$$

$$=1-\frac{2}{n+2}=\frac{n}{n+2}$$

13 $\frac{2}{(r+1)(r+3)}=\frac{A}{r+1}+\frac{B}{r+3}$

$2=A(r+3)+B(r+1)$

$2=2A=-2B$

Let $f(r)=\frac{1}{r+1}$

$$\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)}=\sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$$

$$=f(1)+f(2)-f(n+1)-f(n+2)$$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}$$

$$=\frac{5(n+2)(n+3)-6(n+3)-6(n+2)}{6(n+2)(n+3)}$$

$$=\frac{n(5n+13)}{6(n+2)(n+3)}$$

Hence $a=5, b=13$ and $c=6$

14 a $\text{LHS} = \frac{r+1}{r+2} - \frac{r}{r+1}$

$$= \frac{(r+1)^2 - r(r+2)}{(r+1)(r+2)}$$

$$= \frac{r^2 + 2r + 1 - r^2 - 2r}{(r+1)(r+2)}$$

$$= \frac{1}{(r+1)(r+2)}$$

$= \text{RHS}$, as required

To show that an algebraic identity is true, you should start from one side of the identity, here the left hand side (LHS), and use algebra to show that it is equal to the other side of the identity, here the right hand side (RHS).

b $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{r+1}{r+2} - \frac{r}{r+1} \right)$

You use the identity that you proved in part **a** to break up each term in the summation into two parts.

$$= \frac{2}{3} - \frac{1}{2}$$

This is the LHS of the identity with $r = 1$.

$$+ \frac{3}{4} - \frac{2}{3}$$

This is the LHS of the identity with $r = 2$.

$$+ \frac{4}{5} - \frac{3}{4}$$

This is the LHS of the identity with $r = 3$.

$$\vdots$$

$$+ \frac{n}{n+1} - \frac{n-1}{n}$$

This is the LHS of the identity with $r = n - 1$.

$$\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{n-1+1}{n-1+2} - \frac{n-1}{n-1+1}$$

$$= \frac{n}{n+1} - \frac{n-1}{n}$$

$$+ \frac{n+1}{n+2} - \frac{n}{n+1}$$

This is the LHS of the identity with $r = n$.

$$= \frac{n+1}{n+2} - \frac{1}{2}$$

$$= \frac{2(n+1) - (n+2)}{2(n+2)} = \frac{2n + 2 - n - 2}{2(n+2)}$$

$$= \frac{n}{2(n+2)}$$

The only terms which have not cancelled one another out are the $-\frac{1}{2}$ in the first line of the summation and the $\frac{n+1}{n+2}$ in the last line.

15 a Let $\frac{2}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$

Multiplying throughout by $(x+1)(x+2)(x+3)$

$2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

Substitute $x = -1$

$2 = A \times 1 \times 2 \Rightarrow A = 1$

Substitute $x = -2$

$2 = B \times -1 \times 1 \Rightarrow B = -2$

Substitute $x = -3$

$2 = C \times -2 \times -1 \Rightarrow C = 1$

Hence

$\mathrm{f}(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}$

When –1 is substituted for x then both $B(x+1)(x+3)$ and $C(x+1)(x+2)$ become zero.

b Using the result in part **a** with $x = r$

$$\sum_{r=1}^{n} \mathrm{f}(r) = \frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$
$$+ \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$
$$+ \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$$
$$\vdots$$
$$+ \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$$
$$+ \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$$
$$+ \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$$
$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{3} + \frac{1}{n+2} - \frac{2}{n+2} + \frac{1}{n+3}$$
$$= \frac{1}{6} - \frac{1}{n+2} + \frac{1}{n+3}$$

You use the partial fractions in part **a** to break up each term in the summation into three parts.

Three terms at the beginning of the summation and three terms at the end have not been cancelled out.

This question asks for no particular form of the answer. You should collect together like terms but, otherwise, the expression can be left as it is. You do not have to express your answer as a single fraction unless the question asks you to do this.

16 a $\dfrac{1}{(r-1)^2}-\dfrac{1}{r^2}=\dfrac{r^2-(r-1)^2}{r^2(r-1)^2}$

$=\dfrac{r^2-(r^2-2r+1)}{r^2(r-1)^2}$

$=\dfrac{2r-1}{r^2(r-1)^2}$

Methods for simplifying algebraic fractions can be found in Chapter 1 of book C3.

b $\displaystyle\sum_{r=2}^{n}\frac{2r-1}{r^2(r-1)^2}=\sum_{r=2}^{n}\left(\frac{1}{(r-1)^2}-\frac{1}{r^2}\right)$

$=\dfrac{1}{1^2}-\dfrac{1}{2^2}$

$+\dfrac{1}{2^2}-\dfrac{1}{3^2}$

$+\dfrac{1}{3^2}-\dfrac{1}{4^2}$

$\vdots$

$+\dfrac{1}{(n-2)^2}-\dfrac{1}{(n-1)^2}$

$+\dfrac{1}{(n-1)^2}-\dfrac{1}{n^2}$

$=\dfrac{1}{1^2}-\dfrac{1}{n^2}=1-\dfrac{1}{n^2}$, as required

This summation starts from $r=2$ and not from the more common $r=1$. It could not start from $r=1$ as $\dfrac{1}{(r-1)^2}$ is not defined for that value.

In this summation all of the terms cancel out with one another except for one term at the beginning and one term at the end.

17 a

$\dfrac{4}{r(r+2)}=\dfrac{A}{r}+\dfrac{B}{r+2}$

$4=A(r+2)+Br$

$4=2A=-2B$

Let $\mathrm{f}(r)=\dfrac{1}{r}$

$\displaystyle\sum_{r=1}^{n}\frac{4}{r(r+2)}=2\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+2}\right)$

$=2(\mathrm{f}(1)+\mathrm{f}(2)-\mathrm{f}(n+1)-\mathrm{f}(n+2))$

$=2\left(1+\dfrac{1}{2}-\dfrac{1}{n+1}-\dfrac{1}{n+2}\right)$

$=\dfrac{3(n+1)(n+2)-2(n+2)-2(n+1)}{(n+1)(n+2)}$

$=\dfrac{n(3n+5)}{(n+1)(n+2)}$

Hence $a=3$ and $b=5$

17 b $\sum_{r=50}^{100} \frac{4}{r(r+2)} = \sum_{r=1}^{100} \frac{4}{r(r+2)} - \sum_{r=1}^{49} \frac{4}{r(r+2)}$

$= \frac{100 \times 305}{101 \times 102} - \frac{49 \times 152}{50 \times 51}$

$= 2.960\ 590\ldots - 2.920\ 784$

$= 0.0398$ (4 d.p.)

$\sum_{r=50}^{100} f(r) = \sum_{r=1}^{100} f(r) - \sum_{r=1}^{49} f(r)$

You find the sum from the 50th to the 100th term by subtracting the sum from the first to the 49th term from the sum from the first to the 100th term.

It is a common error to subtract one term too many, in this case the 50th term. The sum you are finding starts with the 50th term. You must not subtract it from the series – you have to leave it in the series.

18 a $4r^2 - 1 = (2r-1)(2r+1)$

Let

$\frac{2}{4r^2-1} = \frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1}$

Multiply throughout by $(2r-1)(2r+1)$

$2 = A(2r+1) + B(2r-1)$

Substitute $r = \frac{1}{2}$

$2 = 2A \Rightarrow A = 1$

Substitute $r = -\frac{1}{2}$

$2 = -2B \Rightarrow B = -1$

Hence

$\frac{2}{4r^2-1} = \frac{1}{2r-1} - \frac{1}{2r+1}$

This question gives you the option to choose your own method (the question has 'or otherwise') and, as you are given the answer, you could, if you preferred, use the method of mathematical induction.

If the method of differences is used, you begin by factorising $4r^2 - 1$, using the difference of two squares, and then express $\frac{2}{(2r-1)(2r+1)}$ in partial fractions.

$\sum_{r=1}^{n} \frac{2}{4r^2-1} = \sum_{r=1}^{n} \left(\frac{1}{2r-1} - \frac{1}{2r+1} \right)$

$= \frac{1}{1} - \frac{1}{3}$

$+ \frac{1}{3} - \frac{1}{5}$

$+ \frac{1}{5} - \frac{1}{7}$

$\vdots$

$+ \frac{1}{2n-3} - \frac{1}{2n-1}$

$+ \frac{1}{2n-1} - \frac{1}{2n+1}$

$= 1 - \frac{1}{2n+1}$, as required

With $r = 1$,

$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2\times1-1} - \frac{1}{2\times1+1} = \frac{1}{1} - \frac{1}{3}$

With $r = n - 1$,

$\frac{1}{2r-1} - \frac{1}{2r+1} = \frac{1}{2\times(n-1)-1} - \frac{1}{2\times(n-1)+1}$

$= \frac{1}{2n-2-1} - \frac{1}{2n-2+1} = \frac{1}{2n-3} - \frac{1}{2n-1}$

The only terms which are not cancelled out in the summation are the $\frac{1}{1}$ at the beginning and the $-\frac{1}{2n+1}$ at the end.

18 b $\sum_{r=11}^{n} \frac{2}{4r^2-1} = \sum_{r=1}^{20} \frac{2}{4r^2-1} - \sum_{r=1}^{10} \frac{2}{4r^2-1}$

$= \left(1 - \frac{1}{41} - 1 + \frac{1}{21}\right)$

$= -\frac{1}{41} + \frac{1}{21} = \frac{-21+41}{41 \times 21}$

$= \frac{20}{861}$

You find the sum from the 11th to the 20th term by subtracting the sum from the first to the 10th term from the sum from the first to the 20th term.

The conditions of the question require an exact answer, so you must not use decimals.

19 a Using the binomial expansion

$(2r+1)^3 = 8r^3 + 12r^2 + 6r + 1$ **(1)**

$(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$ **(2)**

Subtracting **(2)** from **(1)**

$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$ **(3)**

$A = 24,\ B = 2$

Subtracting the two expansions gives an expression in r^2. This enables you to sum r^2 using the method of differences.

b Using identity **(3)** in part **a**

$\sum_{r=1}^{n}(24r^2+2) = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right)$

$24\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 2 = \sum_{r=1}^{n}\left((2r+1)^3 - (2r-1)^3\right)$

$\sum_{r=1}^{n} 2 = \underbrace{2+2+2+\ldots+2}_{n\text{ times}} = 2n$

It is a common error to write $\sum_{r=1}^{n} 2 = 2$.

$24\sum_{r=1}^{n} r^2 + 2n = \cancel{3^3} - 1^3$

$+ \cancel{5^3} + \cancel{3^3}$

$+ \cancel{7^3} - \cancel{5^3}$

$\vdots$

$+ \cancel{(2n-1)^3} - \cancel{(2n-3)^3}$

$+ (2n+1)^3 - \cancel{(2n-1)^3}$

The expression is $(2r+1)^3 - (2r-1)^3$ with $n-1$ substituted for r. $(2(n-1)+1)^3 - (2(n-1)-1)^3$

$= (2n-1)^3 - (2n-3)^3$

$24\sum_{r=1}^{n} r^2 + 2n = (2n+1)^3 - 1$

$24\sum_{r=1}^{n} r^2 = 8n^2 + 12n^2 + 6n + 1 - 1 - 2n$

$= 8n^3 + 12n^2 + 4n = 4n(2n^2 + 3n + 1)$

$= 4n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^2 = \frac{4n(n+1)(2n+1)}{24} = \frac{1}{6}n(n+1)(2n+1)$, as required.

Summing gives you an equation in $\sum r^2$, which you solve. You then factorise the result to give the answer in the form required by the question.

19 c $(3r-1)^2 = 9r^2 - 6r + 1$

Hence

$$\sum_{r=1}^{40}(3r-1)^2 = 9\sum_{r=1}^{40} r^2 - 6\sum_{r=1}^{40} r + \sum_{r=1}^{40} 1$$

In the formula proved in part **b**, you replace the n by 40.

Using the result in part **b**.

$$9\sum_{r=1}^{40} r^2 = 9 \times \frac{1}{6} \times 40 \times 41 \times 81 = 199\ 260$$

Using the standard result $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$,

$$6\sum_{r=1}^{40} r = 6 \times \frac{40 \times 41}{2} = 4920$$

$$\sum_{r=1}^{40} 1 = 40$$

$\sum_{r=1}^{40} 1 = 40$ is 40 ones added together which is, of course, 40.

Combining these results

$$\sum_{r=1}^{40}(3r-1)^2 = 199\ 260 - 4920 + 40 = 194\ 380$$

20 $\frac{1}{r(r+1)(r+2)} = \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$

$1 = A(r+1)(r+2) + Br(r+2) + Cr(r+1)$

$r = 0 : 1 = 2A$

$r = 1 : 1 = -B$

$r = 2 : 1 = 2C$

Let $f(r) = \frac{1}{r}$

$$\begin{aligned}
\sum_{r=1}^{2n} \frac{1}{r(r+1)(r+2)} &= \frac{1}{2}\sum_{r=1}^{2n}\left(\frac{1}{r} - \frac{1}{r+1} + \frac{1}{r+2} - \frac{1}{r+1}\right) \\
&= \frac{1}{2}(f(1) - f(2n+1) + f(2n+2) - f(2)) \\
&= \frac{1}{2}\left(1 - \frac{1}{2} - \frac{1}{2n+1} + \frac{1}{2n+2}\right) \\
&= \frac{1}{4}\left(1 - \frac{2}{2n+1} + \frac{1}{n+1}\right) \\
&= \frac{1}{4}\left(\frac{(n+1)(2n+1) - 2(n+1) + (2n+1)}{(n+1)(2n+1)}\right) \\
&= \frac{1}{4}\frac{2n^2 + 3n}{(n+1)(2n+1)} = \frac{n(2n+3)}{4(n+1)(2n+1)}
\end{aligned}$$

Hence $a = 2, b = 3$ and $c = 4$

21 a $\text{RHS} = r - 1 + \frac{1}{r} - \frac{1}{r+1}$

$= \frac{(r-1)r(r+1) + (r+1) - r}{r(r+1)}$

$= \frac{r(r^2-1)+1}{r(r+1)}$

$= \frac{r^3 - r + 1}{r(r+1)} = \text{LHS, as required}$

To show that an algebraic identity is true, you should start from one side of the identity, here the right hand side (RHS), and use algebra to show that it is equal to the other side of the identity, here the left hand side (LHS).

b Using the result in part **a**

This summation is broken up into 3 separate summations. Only the third of these uses the method of differences.

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 + \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{1} - \frac{1}{2}$$
$$+ \frac{1}{2} - \frac{1}{3}$$
$$+ \frac{1}{3} - \frac{1}{4}$$
$$\vdots$$
$$+ \frac{1}{n-1} + \frac{1}{n}$$
$$+ \frac{1}{n} - \frac{1}{n+1}$$
$$= 1 + \frac{1}{n+1}$$

In the summation, using the method of differences, all of the terms cancel out with one another except for one term at the beginning and one term at the end.

Combining the three summations

$$\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)} = \frac{n(n+1)}{2} - n + 1 - \frac{1}{n+1}$$
$$= \frac{n(n+1)^2 - 2n(n+1) + 2(n+1) - 2}{2(n+1)}$$
$$= \frac{n^3 + 2n^2 + n - 2n^2 - 2n + 2n + 2 - 2}{2(n+1)}$$
$$= \frac{n^3 + n}{2(n+1)} = \frac{n(n^2+1)}{2(n+1)}$$

To complete the question, you put the results of the three summations over a common denominator and simplify the resulting expression as far as possible.

22 $\frac{2r+3}{r(r+1)} = \frac{A}{r} + \frac{B}{r+1}$

$2r+3 = A(r+1) + Br$

$3 = A, 1 = -B$

$$\frac{2r+3}{r(r+1)}\frac{1}{3^r} = \frac{1}{3^r}\left(\frac{3}{r} - \frac{1}{r+1}\right)$$

$$= \frac{1}{3^{r-1}}\frac{1}{r} - \frac{1}{3^r}\frac{1}{r+1}$$

Let $f(r) = \frac{1}{3^{r-1}}\frac{1}{r}$

$$\sum_{r=1}^{n}\frac{2r+3}{r(r+1)}\frac{1}{3^r} = \sum_{r=1}^{n}\left(f(r) - f(r+1)\right)$$

$$= (f(1) - f(n+1))$$

$$= 1 - \frac{1}{3^n(n+1)}$$

23 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$= 1 - \frac{x^2}{2}$, neglecting terms in x^3 and higher powers

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

$= x$, neglecting terms in x^3 and higher powers

The series of cos x and sin x are both given in the formulae book and may be quoted without proof, unless the question specifically asks for a proof.

$$11\sin x - 6\cos x + 5 = 11x - 6\left(1 - \frac{x^2}{2}\right) + 5$$

$$= 11x - 6 + 3x^2 + 5$$

$$= -1 + 11x + 3x^2$$

You substitute the abbreviated series into the expression and collect together terms.

$A = -1,\ B = 11,\ C = 3$

24 $\text{LHS} = \ln(x^2 - x + 1) + \ln(x+1) - 3\ln x$

$= \ln[(x^2 - x + 1)(x+1)] - \ln x^3$

$= \ln\left(\dfrac{x^3+1}{x^3}\right) = \ln\left(1 + \dfrac{1}{x^3}\right)$

You collect together the three terms of the left hand side (LHS) of the expression into a single logarithm using all three log rules; $\log x + \log y = \log xy$

$\log x - \log y = \log\left(\dfrac{x}{y}\right)$,

and $n\log x = \log x^n$.

$(x^2 - x + 1)(x+1) = x^3 + x^2 - x^2 - x + x + 1$

$= x^3 + 1$

Substituting $\dfrac{1}{x^3}$ for x and n for r in the series

This series is given in the formulae booklet. It is valid for $-1 < x \leqslant 1$ and, if $x > 1$, then $0 < \dfrac{1}{x^3} < 1$ so the series is valid for this question.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{r+1}x^t}{r} + \ldots$$

$$\text{LHS} = \frac{1}{x^3} - \frac{1}{2x^6} + \ldots + \frac{(-1)^{n-1}}{nx^{3n}} + \ldots, \text{ as required}$$

$(-1)^{n+1} = (-1)^{n-1}$. If n is odd, both sides are 1. If n is even, both sides are -1.

25 $e^{-2x} = 1 + (-2x) + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \ldots$

$= 1 - 2x + 2x^2 - \dfrac{4}{3}x^3 + \ldots$

Substituting $-2x$ for x in the formula $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$ and ignoring terms in x^4 and higher powers.

$\cos 5x = 1 - \dfrac{(5x)^2}{2!} + \ldots$

$= 1 - \dfrac{25}{2}x^2 + \ldots$

Substituting $5x$ for x in the formula $\cos x = 1 - \dfrac{x^2}{2!} + \ldots$ and ignoring terms in x^4 and higher powers.

$e^{-2x}\cos 5x = \left(1 - 2x + 2x^2 - \dfrac{4}{3}x^3 + \ldots\right)\left(1 - \dfrac{25}{2}x^2 + \ldots\right)$

$= 1 - \dfrac{25}{2}x^2 - 2x + 25x^3 + 2x^2 - \dfrac{4}{3}x^3 + \ldots$

$= 1 - 2x + \left(-\dfrac{25}{2} + 2\right)x^2 + \left(25 - \dfrac{4}{3}\right)x^3 + \ldots$

$= 1 - 2x - \dfrac{21}{2}x^2 + \dfrac{71}{3}x^3 + \ldots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $2x^2$ by $-\dfrac{25}{2}x^2$ gives $-25x^4$ and you just ignore his term.

$A = 1,\ B = -2,\ C = -\dfrac{21}{2}, D = \dfrac{71}{3}$

26 a $(2x+3)^{-1} = 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$

Part **a** is a binomial series with a rational index.

$$= \frac{1}{3}\left(1 - \frac{2x}{3} + \frac{(-1)(-2)}{2.1}\left(\frac{2x}{3}\right)^2 + \frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{2x}{3}\right)^3 + \ldots\right)$$

$$= \frac{1}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 + \ldots\right)$$

$$= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots$$

b $\frac{\sin 2x}{3+2x} = \sin 2x(3+2x)^{-1}$

$$= \left(2x - \frac{(2x)}{3!} + \ldots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots\right)$$

$$= \left(2x - \frac{4}{3}x^3 + \ldots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots\right)$$

$$= \frac{2}{3}x - \frac{4}{9}x^2 + \frac{8}{27}x^3 - \frac{16}{81}x^4 - \frac{4}{9}x^3 + \frac{8}{27}x^4 + \ldots$$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $-\frac{4}{3}x^3$ by $\frac{4}{27}x^2$ gives $-\frac{16}{81}x^5$ and you ignore this term.

$$= \frac{2}{3}x - \frac{4}{9}x^2 + \left(\frac{8}{27} - \frac{4}{9}\right)x^3 + \left(\frac{8}{27} - \frac{16}{81}\right)x^4 + \ldots$$

$$= \frac{2}{3}x - \frac{4}{9}x^2 - \frac{4}{27}x^3 + \frac{8}{81}x^4 + \ldots$$

27 a $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$$= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) \quad \textbf{(1)},$$

neglecting terms above x^4

The expression $-\frac{x^2}{2!} + \frac{x^4}{4!}$ is used to replace the x in the standard series for $\ln(1+x)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \ldots$$

Using the expansion **(1)**

$$\ln(\cos x) = \ln\left(1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \ldots$$

$-\frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 = -\frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{1152}$

but, as the expansion is only required up to the term in x^4, you only need the first of the three terms.

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + \ldots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \ldots$$

27 b $\ln(\sec x) = \ln\left(\frac{1}{\cos x}\right) = \ln 1 - \ln\cos x$

$= -\ln\cos x$

Using the log rule $\log\left(\frac{a}{b}\right) = \log a - \log b$ and the fact that $\ln 1 = 0$.

Using the result to part **a**

$$\ln(\sec x) = -\left(-\frac{x^2}{2} - \frac{x^4}{12} - \dots\right) = \frac{x^2}{2} + \frac{x^4}{12} + \dots$$

28 a Let $u = 1 + \cos 2x$, then $f(x) = \ln u$

$$\frac{du}{dx} = -2\sin 2x$$

$$f'(x) = f'(u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{1+\cos 2x} \times -2\sin 2x$$

$$= \frac{-4\sin x\cos x}{2\cos^2 x}$$

Using the identities $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$.

$$= \frac{-2\sin x}{\cos x} = -2\tan x, \text{ as required}$$

b $f''(x) = -2\sec^2 x$

$f'''(x) = -4\sec^2 x\tan x$

$f''''(x) = -8\sec x \cdot \sec x\tan x \cdot \tan x - 4\sec^2 x \cdot \sec^2 x$

$= -8\sec^2 x\tan^2 x - 4\sec^4 x$

$= -[-4\sec^2 x\tan x \times -2\tan x + (-2\sec^2 x)^2]$

$= -[f'''(x)f'(x) + (f''(x))^2]$, as required

$F''''(x)$ is a symbol used for the fourth derivative of $f(x)$ with respect to x. The symbol $f^{(iv)}(x)$ is also used for the fourth derivative.

You use the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = -4\sec^2 x$ and $v = \tan x$. You also use the chain rule

$\frac{d}{dx}(\sec^2 x) = 2\sec x\frac{d}{dx}(\sec x)$

$= 2\sec x \times \sec x\tan x$.

c $f(0) = \ln(1 + \cos 0) = \ln 2$

$f'(0) = -2\tan 0 = 0$

$f''(0) = -2\sec^2 0 = -2$

$f'''(0) = -4\sec^2 0\tan 0 = 0$

$f''''(0) = -[f'''(0)f'(0) + (f''(0))^2]$

Using the result for part **b**.

$= -[0 \times 0 + (-2)^2] = -4$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \dots$$

$$= \ln + 2 + x \times 0 + \frac{x^2}{2} \times -2 + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times -4 + \dots$$

$$= \ln 2 - x^2 - \frac{1}{6}x^4 + \dots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

29 $\int_0^\infty e^{-x}\sin x\mathrm{d}x = \left[-e^{-x}\cos x\right]_0^\infty - \int_0^\infty e^{-x}\cos x\mathrm{d}x$

$\int_0^\infty e^{-x}\cos x\mathrm{d}x = \left[e^{-x}\sin x\right]_0^\infty + \int_0^\infty e^{-x}\sin x\mathrm{d}x$

$\int_0^\infty e^{-x}\sin x\mathrm{d}x = \left[-e^{-x}\cos x\right]_0^\infty$

$-\left[e^{-x}\sin x\right]_0^\infty - \int_0^\infty e^{-x}\sin x\mathrm{d}x$

$2\int_0^\infty e^{-x}\sin x\mathrm{d}x = \left[-e^{-x}\cos x - e^{-x}\sin x\right]_0^\infty = 1$

$\int_0^\infty e^{-x}\sin x\mathrm{d}x = \frac{1}{2}$

30 $\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x = \left[-x\sqrt{1-x^2}\right]_0^1 + \int_0^1 \sqrt{1-x^2}\mathrm{d}x$

$\left(u = x, u' = 1, v' = \frac{x}{\sqrt{1-x^2}}, v = -\sqrt{1-x^2}\right)$

$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x = \int_0^1 -\frac{1-x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$= \left[\arcsin x\right]_0^1 + \int_0^1 \sqrt{1-x^2}\mathrm{d}x$

$2\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x = \left[\arcsin x - x\sqrt{1-x^2}\right]_0^1$

$= \arcsin 1 = \frac{\pi}{2}$

$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{4}$

31 a $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$

$1 = A(x+3) + Bx, 1 = 3A = -3B$

$\int \frac{1}{x(x+3)}\mathrm{d}x = \frac{1}{3}\int \frac{1}{x} - \frac{1}{x+3}\mathrm{d}x$

$= \frac{1}{3}(\ln x - \ln(x+3)) = \frac{1}{3}\ln\frac{x}{x+3} + c$

b $\int_3^\infty \frac{1}{x(x+3)}\mathrm{d}x = \lim_{t\to\infty}\int_3^t \frac{1}{x(x+3)}\mathrm{d}x$

$= \lim_{t\to\infty}\left(\frac{1}{3}\ln\frac{t}{t+3} - \frac{1}{3}\ln\frac{3}{3+1}\right)$

$= \frac{1}{3}\ln 1 - \frac{1}{3}\ln\frac{1}{2} = \frac{1}{3}\ln 2$

32 $\int_1^{\infty} x^3 e^{-x^4} \mathrm{d}x = \lim_{t\to\infty} \int_1^t x^3 e^{-x^4} \mathrm{d}x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^4}\right) = -4x^3 e^{-x^4}$$

$$\int_1^t x^3 e^{-x^4} \mathrm{d}x = -\frac{1}{4}\left[e^{-x^4}\right]_1^t = \frac{1}{4}\left(e^{-1} - e^{-t^4}\right)$$

$$\int_1^{\infty} x^3 e^{-x^4} \mathrm{d}x = \lim_{t\to\infty} \frac{1}{4}\left(e^{-1} - e^{-t^4}\right) = \frac{1}{4e}$$

33 a $\int \frac{1}{(5-2x)^2} \mathrm{d}x = \frac{1}{2}\int 2(5-2x)^{-2} \mathrm{d}x = \frac{1}{2(5-2x)} + c$

b $\int_{-\infty}^{3} \frac{1}{(5-2x)^2} \mathrm{d}x = \lim_{t\to\frac{5}{2}} \int_t^3 \frac{1}{(5-2x)^2} \mathrm{d}x + \lim_{s\to\frac{5}{2}} \int_{-\infty}^{s} \frac{1}{(5-2x)^2} \mathrm{d}x$

$$\int_{-\infty}^{s} \frac{1}{(5-2x)^2} \mathrm{d}x = \frac{1}{2}\frac{1}{(5-2s)} \to \infty \text{ as } s \to \frac{5}{2}$$

34 $\mathrm{f}(x) = x\cos 2x$

$$\int \mathrm{f}(x)\mathrm{d}x = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x \mathrm{d}x$$

$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

$$\bar{\mathrm{f}} = \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \mathrm{f}(x)\mathrm{d}x = \frac{2}{\pi}\left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\frac{1}{4}\cos\pi - \frac{1}{4}\cos 0\right) = \frac{2}{\pi}\left(-\frac{1}{2}\right) = -\frac{1}{\pi}$$

35 a $\mathrm{f}(x) = \frac{3x}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$

$$3x = A(2x-3) + B(x-1), 3 = -A, 0 = -3A - B$$

$$\mathrm{f}(x) = -3\left(\frac{1}{x-1} - \frac{3}{2x-3}\right)$$

$$\bar{\mathrm{f}} = \frac{1}{5-2}\int_2^5 -3\left(\frac{1}{x-1} - \frac{3}{2x-3}\right)\mathrm{d}x$$

$$= \int_2^5 -\frac{1}{x-1} + \frac{3}{2x-3}\mathrm{d}x$$

$$= \left[-\ln(x-1) + \frac{3}{2}\ln(2x-3)\right]_2^5$$

$$= \left(-\ln 4 + \ln 1 + \frac{3}{2}\ln 7 - \frac{3}{2}\ln 1\right)$$

$$= \frac{1}{2}(\ln 7^3 - \ln 4^2) = \frac{1}{2}\ln\frac{343}{16}$$

35 b $\mathrm{f}(x) + \ln k$ has mean value $\bar{\mathrm{f}} + \ln k$

$= \frac{1}{2}\ln\frac{343}{16} + \ln k = \frac{1}{2}\ln\frac{343k^2}{16}$

36 a $\mathrm{f}(x) = x^2(x^3 - 1)^3$

$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 1)^4 = 4(x^3 - 1)^3(3x^2) = 12x^2(x^3 - 1)^3$

$\bar{\mathrm{f}} = \frac{1}{3-1}\int_1^3 x^2(x^3 - 1)^3\,\mathrm{d}x = \frac{1}{24}\int_1^3 12x^2(x^3 - 1)^3\,\mathrm{d}x$

$= \frac{1}{24}\left[(x^3 - 1)^4\right]_1^3 = \frac{57122}{3}$

b $k\mathrm{f}(x)$ has mean value $k\bar{\mathrm{f}}$ over $[1, 3]$

so $-2\mathrm{f}(x)$ has mean value $-\frac{114244}{3}$

37 $\bar{\mathrm{f}} = \frac{1}{3-1}\int_1^3 \frac{1}{\sqrt{3-x}}\,\mathrm{d}x = -\int_1^3 -\frac{1}{2\sqrt{3-x}}\,\mathrm{d}x$

$= -\left[\sqrt{3-x}\right]_1^3 = \sqrt{2}$

38 $\mathrm{f}(x) = \ln kx$

$\bar{\mathrm{f}} = \frac{1}{5-1}\int_1^5 \ln kx\,\mathrm{d}x$

$\int \ln kx\,\mathrm{d}x = x\ln kx - \int 1\,\mathrm{d}x = x(\ln kx - 1) + c$

$\bar{\mathrm{f}} = \frac{1}{4}\left[x(\ln kx - 1)\right]_1^5 = \frac{1}{4}(5(\ln 5k - 1) - \ln k + 1)$

$= \frac{1}{4}\ln\frac{(5k)^5}{k} - 1 = \frac{1}{4}\ln 5^5k^4 - 1 = \frac{1}{4}\ln 5^9 - 1$

$k = 5$

39 a $y = (\arcsin x)^2$

Let $u = \arcsin x$

$$y = u^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{du} = 2u$$

$$\frac{du}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

This result is in the Edexcel formula booklet, which is provided for use with the paper. It is a good idea to quote any formulae you use in your solution.

Hence

$$\frac{dy}{dx} = 2u \times \frac{1}{\sqrt{(1-x^2)}} = \frac{2\arcsin x}{\sqrt{(1-x^2)}}$$

$$\sqrt{(1-x^2)}\frac{dy}{dx} = 2\arcsin x$$

Square both sides of this solution and use the given $y = (\arcsin x)^2$ to complete the solution.

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\arcsin x)^2$$

$$= 4y, \text{ as required}$$

b Differentiating the result of part **a** implicitly with respect to x

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4\frac{dy}{dx}$$

Using the chain rule $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) = 2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}$.

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = 2$$

Divide the equation throughout by $2\frac{dy}{dx}$.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2, \text{ as required}$$

40 a $y = \arctan 3x$

$\tan y = 3x$

Differentiating implicitly with respect to x

$$\sec^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\sec^2 y} = \frac{3}{1 + \tan^2 y}$$

$$= \frac{3}{1+9x^2}, \text{ as required}$$

40 b Using integration by parts and the result in part **a**

You use $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ with u = artanh $3x$ and $\frac{dv}{dx} = 6x$. You know $\frac{du}{dx}$ from part **a**.

$$\int 6x \arctan 3x\, dx = 3x^2 \arctan 3x - \int 3x^2 \times \frac{3}{1+9x^2} dx$$

$$= 3x^2 \arctan 3x - \int \frac{9x^2+1-1}{1+9x^2} dx$$

$$= 3x^2 \arctan 3x - \int 1\, dx + \int \frac{1}{1+9x^2} dx$$

$$= 3x^2 \arctan 3x - x + \frac{1}{3}\arctan 3x$$

You have to integrate $\frac{9x^2}{1+9x^2}$. As the degree of the numerator is equal to the degree of the denominator, you must divide the denominator into the numerator before integrating.

The adaptation of the formula given in the Edexcel formulae booklet, $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan\left(\frac{x}{a}\right)$ to this integral is not straightforward.

$\int \frac{1}{1+9x^2} dx = \frac{1}{9}\int \frac{1}{\frac{1}{9}+x^2} dx$

$= \frac{1}{9}\times\frac{1}{\frac{1}{3}}\arctan\left(\frac{x}{\frac{1}{3}}\right) = \frac{1}{3}\arctan 3x$. You may prefer to find such an integral using the substitution $3x = \tan\theta$.

$$\left[3x^2 \arctan 3x - x + \frac{1}{3}\arctan 3x\right]_0^{\frac{\sqrt{3}}{3}}$$

$$= 3\times\left(\frac{\sqrt{3}}{3}\right)^2 \arctan\sqrt{3} - \frac{\sqrt{3}}{3} + \frac{1}{3}\arctan\sqrt{3}$$

$$= \frac{4}{3}\arctan\sqrt{3} - \frac{\sqrt{3}}{3}$$

$$= \frac{4}{3}\times\frac{\pi}{3} - \frac{\sqrt{3}}{3} = \frac{1}{9}(4\pi - 3\sqrt{3}), \text{ as required}$$

41 a Let $y = f(x) = \arcsin x$

$\sin y = x$

Differentiating implicitly with respect to x

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sinh^2)y}} = \frac{1}{\sqrt{(1-x^2)}}$$

$$f'(x) = \frac{1}{\sqrt{(1-x^2)}}, \text{ as required}$$

Unless otherwise stated, arcsin x is taken to have the range $-\frac{\pi}{2} < \arcsin x < \frac{\pi}{2}$. These are the principal values of arcsin x. In this range, arcsin x is an increasing function of x, $\frac{dy}{dx}$ is positive and you can take the positive value of the square root.

41 b $y = \arcsin 2x$

Let $u = 2x, \dfrac{du}{dx} = 2$

$y = \arcsin u$

Using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{\sqrt{(1-u^2)}} \times 2 = \frac{2}{\sqrt{(1-4x^2)}}$$

c $x = \dfrac{1}{2}\sin\theta \Rightarrow \dfrac{dx}{d\theta} = \dfrac{1}{2}\cos\theta$

In this question it is convenient to carry out the substitution without returning to the original variable x. So at some stage you must change the x limits to θ limits.

At $x = \dfrac{1}{4}, \dfrac{1}{4} = \dfrac{1}{2}\sin\theta \Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$

At $x = 0, 0 = \dfrac{1}{2}\sin\theta \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$$\int \frac{x\arcsin 2x}{\sqrt{(1-4x^2)}}\,dx = \int \frac{\frac{1}{2}\sin\theta\arcsin(\sin\theta)}{\sqrt{(1-\sin^2\theta)}}\left(\frac{dx}{d\theta}\right)d\theta$$
$$= \int \frac{\frac{1}{2}\sin\theta \times \theta}{\cos\theta}\left(\frac{1}{2}\cos\theta\right)d\theta$$

By definition, $\arcsin(\sin\theta) = \theta$.

$$= \frac{1}{4}\int \theta\sin\theta\,d\theta$$

You use integration by parts, $\int u\dfrac{dv}{d\theta} = uv - \int v\dfrac{du}{d\theta}d\theta$, with $u = \theta$ and $\dfrac{dv}{d\theta} = \sin\theta$.

$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\int\cos\theta\,d\theta$$
$$= -\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta$$

Hence

$$\int_0^{\frac{1}{4}} \frac{x\arcsin 2x}{\sqrt{(1-4x^2)}}\,dx = \left[-\frac{1}{4}\theta\cos\theta + \frac{1}{4}\sin\theta\right]_0^{\frac{\pi}{6}}$$
$$= \left[-\frac{\pi}{24}\cos\frac{\pi}{6} + \frac{1}{4}\sin\frac{\pi}{6}\right] - [0]$$
$$= -\frac{\pi}{24} \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{1}{2}$$
$$= \frac{1}{48}\left(6 - \pi\sqrt{3}\right), \text{ as required}$$

42 $\dfrac{2x+1}{x(x^2+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x}$

$2x+1 = (Ax+B)x + C(x^2+1)$

Coefficients of $x^2 : 0 = A + C,$

$x : 2 = B,$

$1 : 1 = C$ so $A = -1, B = 2, C = 1$

$$\int \frac{2x+1}{x(x^2+1)}\,dx = \int \frac{2}{x^2+1} - \frac{x}{x^2+1} + \frac{1}{x}\,dx$$

$$= 2\arctan x - \frac{1}{2}\ln(x^2+1) + \ln x + c$$

Hence $A = 2$ and $B = -\dfrac{1}{2}$

43 a $f(x) = \dfrac{3x^2+5x}{x^3-3x^2+5x-15} = \dfrac{3x^2+5x}{(x-3)(x^2+5)}$

$= \dfrac{A}{x-3} + \dfrac{Bx+C}{x^2+5}$

$3x^2+5x = A(x^2+5) + (Bx+C)(x-3)$

$x = 3 : 42 = 14A$

Coefficients of $x^2 : 3 = A + B,$

$x : 5 = C - 3B,$

$1 : 0 = 5A - 3C$ so $A = 3, B = 0, C = 5$

$f(x) = \dfrac{3}{x-3} + \dfrac{5}{x^2+5}$

b $\displaystyle\int f(x)\,dx = \int \frac{3}{x-3} + \frac{5}{x^2+5}\,dx$

$$= 3\ln(x-3) + 5\frac{1}{\sqrt{5}}\arctan\frac{x}{\sqrt{5}} + c$$

$$= 3\ln(x-3) + \sqrt{5}\arctan\frac{x}{\sqrt{5}} + c$$

Hence $P = 3, Q = \sqrt{5}$ and $R = \dfrac{1}{\sqrt{5}}$

44 Use

$$V = \pi \int y^2 \, dx$$

$$= \pi \int_1^3 x^2 e^{2x} \, dx$$

Let $u = x^2$ and $\frac{dv}{dx} = e^{2x}$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$v = \frac{1}{2}e^{2x} \Leftarrow \frac{dv}{dx} = e^{2x}$$

Complete the table for u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$. Take care to differentiate u but integrate $\frac{dv}{dx}$.

$$\therefore V = \pi\left[x^2 \cdot \frac{1}{2}e^{2x}\right]_1^3 - \pi\int_1^3 \frac{1}{2}e^{2x} \cdot 2x \, dx.$$

$$\text{i.e. } V = \pi\left[\frac{9}{2}e^6 - \frac{1}{2}e^2\right] - \pi\int_1^3 xe^{2x} \, dx.$$

This integral is simpler than V but still not one you can write down. Use integration by parts again with $u = x$ and $\frac{dv}{dx} = e^{2x}$.

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$v = \frac{1}{2}e^{2x} \Rightarrow \frac{dv}{dx} = e^{2x}$$

Complete a new table for the new u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$.

$$\therefore V = \pi\left[\frac{9}{2}e^6 - \frac{1}{2}e^2\right] - \pi\left[x \cdot \frac{1}{2}e^{2x}\right]_1^3 + \pi\int_1^3 \frac{1}{2}e^{2x} \cdot 1 \, dx$$

Apply the integration by parts formula a second time.

$$= \pi\left[\frac{9}{2}e^6 - \frac{1}{2}e^2\right] - \pi\left[\frac{3}{2}e^6 - \frac{1}{2}e^2\right] + \pi\left[\frac{1}{4}e^{2x}\right]_1^3$$

$$= \frac{13}{4}\pi e^6 - \frac{\pi}{4}e^2$$

45 a $\text{Area} = \int_0^{2\pi} 3\sin\frac{x}{2} \, dx$

Recall (5) in the introduction to integrating. Integration a sin function gives a change of sign and a cos function.

$$= \left[3 \times -2\cos\frac{x}{2}\right]_0^{2\pi}$$

The 2 here is obtained from dividing by $\frac{1}{2}$ which anses from the chain rule.

$$= \left[-6\cos\frac{x}{2}\right]_0^{2\pi}$$

$$= [6 - (-6)]$$

Area = 12

45 b Volume $= \pi \int_0^{2\pi} 9\sin^2 \frac{x}{2} \, dx$

> You cannot integrate $\sin^2 \frac{x}{2}$, but you can write this in terms of $\cos x$.

Recall $\cos 2A = 1 - 2\sin^2 A$

So $\sin^2 A = \frac{1}{2}(1 - 2\cos 2A)$

So $\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x)$

$\therefore$ Volume $= \frac{9\pi}{2} \int_0^{2\pi} (1 - \cos x) \, dx$

> You can now integrate each term directly.

$= \frac{9\pi}{2} [x + \sin x]_0^{2\pi}$

$= \frac{9\pi}{2} \times (2\pi - 0)$

volume $= 9\pi^2$

46 Volume $= \pi \int_0^{\pi} (x\sqrt{\sin x})^2 \, dx$

> Use $v = \pi \int y^2 \, dx$.

$= \pi \int_0^{\pi} x^2 \sin x \, dx$

Use integration by parts.

> Let $u = x^2$ and $\frac{dv}{dx} = \sin x$.

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$v = -\cos x \Leftarrow \frac{dv}{dx} = \sin x$

> Complete the table for $u, v, \frac{du}{dx}$ and $\frac{dv}{dx}$.

$\therefore$ Volume $= \pi \left[[-x^2 \cos x]_0^{\pi} - \int_0^{\pi} -2x \cos x \, dx \right]$

$= \pi \left(\pi^2 + \int_0^{\pi} 2x \cos x \, dx \right)$

> This integral is simpler than the original one but you will need to use integration by parts again, with $u = 2x$ and $\frac{dv}{dx} = \cos x$.

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$v = \sin x \Leftarrow \frac{dv}{dx} = \cos x$

$\therefore$ Volume $= \pi \left[\pi^2 + [2x \sin x]_0^{\pi} - \int_0^{\pi} 2\sin x \, dx \right]$

> This term becomes zero as $2\pi \sin \pi - 0 = 0$.

$= \pi \left(\pi^2 + [2\cos x]_0^{\pi} \right)$

$= \pi \left(\pi^2 + [-2 - 2] \right)$

$= \pi(\pi^2 - 4)$

$= \pi^3 - 4\pi$

47 $V = \pi\int_a^5 x^2 dy = \pi\int_a^5 \frac{1}{(3y-1)^2}\,dy$

$= \pi\left[-\frac{1}{3(3y-1)}\right]_a^5 = \pi\left(-\frac{1}{42}+\frac{1}{3(3a-1)}\right) = \frac{3\pi}{70}$

$-5+\frac{70}{3a-1} = 9,\ 14 = \frac{70}{3a-1}$

$42a = 84,\ a = 2$

48 a $x = y\cos y$

$x = 0$ when $y\cos y = 0$

$y = 0$ or $\cos y = 0, y = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$k = \frac{\pi}{2}$

b $V = \pi\int_0^{\frac{\pi}{2}} x^2 dy = \pi\int_0^{\frac{\pi}{2}} y^2\cos^2 y\,dy$

$= \pi\int_0^{\frac{\pi}{2}} y^2\left(\frac{1}{2}(1+\cos 2y)\right)dy$

$u = y^2, u' = 2y, v' = \frac{1}{2}(1+\cos 2y), v = \frac{1}{2}y+\frac{1}{4}\sin 2y$

$V = \pi\left[\frac{1}{2}y^3+\frac{1}{4}y^2\sin 2y\right]_0^{\frac{\pi}{2}} - \pi\int_0^{\frac{\pi}{2}} y^2+\frac{1}{2}y\sin 2y\,dy$

$\int_0^{\frac{\pi}{2}} y^2+\frac{1}{2}y\sin 2y\,dy = \left[\frac{y^3}{3}-\frac{1}{4}y\cos 2y\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{1}{4}\cos 2y\,dy$

$V = \pi\left[\frac{y^3}{2}+\frac{y^2}{4}\sin 2y-\frac{y^3}{3}+\frac{1}{4}y\cos 2y+\frac{1}{8}\sin 2y\right]_0^{\frac{\pi}{2}}$

$= \pi\left[\frac{y^3}{6}+\left(\frac{y^2}{4}+\frac{1}{8}\right)\sin 2y+\frac{1}{4}y\cos 2y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi^4}{48}+\frac{\pi^2}{8}\cos\pi = \frac{\pi^4}{48}-\frac{\pi^2}{8}$

Hence $a = \frac{1}{48}$ and $b = -\frac{1}{8}$

49 $V=\pi\int_{f^{-1}(\ln 2)}^{f^{-1}(b)} y^2\frac{dx}{dt}dt$, let $b=\ln a$

$V=\pi\int_2^a (t^2-2)\frac{1}{t}dt=\pi\int_2^a t^3-4t+\frac{4}{t}dt$

$=\pi\left[\frac{t^4}{4}-2t^2+4\ln t\right]_2^a=\pi\left(\frac{a^4}{4}-2a^2+4\ln a+4-4\ln 2\right)$

$=\pi\left(2a^2\left(\frac{a^2}{8}-1\right)+4+4\ln\frac{a}{2}\right)$

$a=4, V=\pi(36+4\ln 2)$

$b=\ln a=\ln 4$

50 a $V=\pi\int_0^4 y^2dx=\pi\int_0^4\frac{25}{1+4x}dx$

$=\pi\left[\frac{25}{4}\ln(1+4x)\right]_0^4=\frac{25\pi}{4}\ln 17$

Volume scale factor is $(2\text{cm})^3$

$V=\frac{25}{4}\ln 17\times 8\text{cm}^3=50\pi\ln 17\text{cm}^3$

b Non-zero thickness of the vase

51 a $x=\sqrt{y}\left(3-\frac{1}{2}e^y\right)$

$x=0\Rightarrow y=0$ or $3-\frac{1}{2}e^y=0, e^y=6$

Intersection at $(0,\ln 6)$

b $V=\pi\int_0^{\ln 6}x^2dy=\pi\int_0^{\ln 6}y\left(3-\frac{1}{2}e^y\right)^2dy$

$=\pi\int_0^{\ln 6}y\left(9-3e^y+\frac{1}{4}e^{2y}\right)dy$

$\int ye^{ay}dy=\frac{1}{a}ye^{ay}-\int\frac{1}{a}e^{ay}dy=\frac{1}{a^2}e^{ay}(ay-1)$

$V=\pi\int_0^{\ln 6}9y-3ye^y+\frac{1}{4}ye^{2y}dy$

$=\pi\left[\frac{9y^2}{2}-3ye^y+3e^y+\frac{1}{4}\left(\frac{1}{2}ye^{2y}-\frac{1}{4}e^{2y}\right)\right]_0^{\ln 6}$

$=\pi\left(\frac{9(\ln 6)^2}{2}-18\ln 6+18+\frac{18}{4}\ln 6-\frac{36}{6}-\left(3-\frac{1}{16}\right)\right)$

$=\pi\left(\frac{9(\ln 6)^2}{2}-\frac{27}{2}\ln 6+\frac{205}{16}\right)$

Volume scale factor is $(1\text{cm})^3$

$V=9.65\text{cm}^3$ (2 d.p.)

51 c Filament may be wasted

52 a $y = 0 = 3\sin 2t, 2t = \pi$

$$V = \pi\int_0^{\frac{\pi}{2}} y^2 \frac{dx}{dt} dt = \pi\int_0^{\frac{\pi}{2}} (3\sin 2t)^2 (2\cos t)dt$$

Volume scale factor is $(0.5\text{cm})^3$

$$V = \frac{9\pi}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t \cos t dt \text{ cm}^3$$

$$= \frac{9\pi}{4}\int_0^{\frac{\pi}{2}} 4\sin^2 t \cos^2 t \cos t dt \text{ cm}^3$$

$$= 9\pi\int_0^{\frac{\pi}{2}} \sin^2 t \cos^3 t dt \text{ cm}^3$$

b $V = 9\pi\int_0^{\frac{\pi}{2}} \sin^2 t \cos^3 t dt \text{ cm}^3$

$$= 9\pi\int_0^{\frac{\pi}{2}} \sin^2 t(1-\sin^2 t)\cos t dt \text{ cm}^3$$

$$= 9\pi\left[\frac{\sin^3 t}{3} - \frac{\sin^5 t}{5}\right]_0^{\frac{\pi}{2}} \text{ cm}^3$$

$$= 9\pi\left(\frac{1}{3} - \frac{1}{5}\right)\text{cm}^3 = \frac{6\pi}{5}\text{cm}^3$$

Challenge

1 a $\omega = e^{\frac{2\pi i}{3}}, \omega^{3k} = e^{2\pi ik} = 1$

$$n = 0: \frac{1^n + \omega^n + \left(\omega^2\right)^n}{3} = \frac{1+1+1}{3} = 1$$

$$n = 3k: \frac{1^{3k} + \omega^{3k} + \left(\omega^2\right)^{3k}}{3} = \frac{1^{3k} + \omega^{3k} + \left(\omega^{3k}\right)^2}{3}$$

$$= \frac{1+1+1}{3} = 1$$

$$n = 3k+1: \frac{1^{3k+1} + \omega^{3k+1} + \left(\omega^2\right)^{3k+1}}{3} = \frac{1 + \omega^{3k+1} + \omega^{6k+2}}{3}$$

$$= \frac{1 + \omega^{3k}\omega + \left(\omega^{3k}\right)^2\omega^2}{3} = \frac{1+\omega+\omega^2}{3} = 0$$

$$n = 3k+2: \frac{1^{3k+1} + \omega^{3k+2} + \left(\omega^2\right)^{3k+2}}{3} = \frac{1 + \omega^{3k+2} + \omega^{6k+4}}{3}$$

$$= \frac{1 + \omega^{3k}\omega^2 + \left(\omega^{3k}\right)^2\omega^3\omega}{3} = \frac{1+\omega^2+\omega}{3} = 0$$

Challenge

1 b $\mathrm{f}(x)=\sum a_n x^n$

$\mathrm{f}(1)=\sum a_n, \mathrm{f}(\omega)=\sum a_n\omega^n, \mathrm{f}(\omega^2)=\sum a_n\left(\omega^2\right)^n$

$$\frac{f(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\sum a_n\frac{1^n+\omega^n+\left(\omega^2\right)^n}{3}$$

$$=\sum a_n 1_{\{n=3k\}},$$

where $1_{\{n=3k\}}=1$ if $n=3k$ and 0 otherwise

c $\mathrm{f}(x)=(1+x)^{45}=\sum_{r=0}^{45}\binom{45}{r}x^r$

$$S=\sum_{r=0}^{45}\binom{45}{r}1_{\{r=3k\}}=\sum_{r=0}^{15}\binom{45}{3r}$$

$$=\frac{\mathrm{f}(1)+\mathrm{f}(\omega)+\mathrm{f}(\omega^2)}{3}=\frac{2^{45}+(1+\omega)^{45}+(1+\omega^2)^{45}}{3}$$

$$=\frac{2^{45}+(-\omega^2)^{45}+(-\omega)^{45}}{3}=\frac{2^{45}-(\omega^3)^{30}-(\omega^3)^{15}}{3}$$

$$=\frac{2^{45}-2}{3}\text{ as }1+\omega+\omega^2=0\text{ and }\ \omega^3=0$$

2 $\mathrm{f}(x)=\sqrt{\dfrac{x+3}{x^3}}=y$

$$V=\pi\int_2^\infty y^2dx=\pi\lim_{t\to\infty}\int_2^t\frac{x+3}{x^3}\mathrm{d}x$$

$$=\pi\lim_{t\to\infty}\int_2^t\frac{1}{x^2}+\frac{3}{x^3}\mathrm{d}x=\pi\lim_{t\to\infty}\left[-\frac{1}{x}-\frac{3}{2x^2}\right]_2^t$$

$$=\pi\lim_{t\to\infty}\left(-\frac{1}{t}-\frac{3}{2t^2}+\frac{1}{2}+\frac{3}{8}\right)=\frac{7\pi}{8}$$

3 a $\mathrm{f}(x)=\dfrac{A}{1+x^2}$

$$\int_{-\infty}^{\infty}\mathrm{f}(x)\mathrm{d}x=A\int_{-\infty}^{\infty}\frac{1}{1+x^2}\mathrm{d}x=A\left[\arctan x\right]_{-\infty}^{\infty}$$

$$=A\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=A\pi=1, A=\frac{1}{\pi}$$

b $$\int_{-\infty}^{\infty}x^2\mathrm{f}(x)\mathrm{d}x=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x^2}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{\pi}\lim_{s,t\to\infty}\int_{-s}^t 1-\frac{1}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{\pi}\lim_{s,t\to\infty}\left[x-\arctan x\right]_{-s}^t\to\infty$$

Challenge

3 c $$\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,dx=\lim_{a\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_{-a}^{a}$$

$$=\lim_{a\to\infty}\left(\frac{1}{2}\ln\left(1+a^2\right)-\frac{1}{2}\ln\left(1+(-a)^2\right)\right)=0$$

$$\lim_{a\to\infty}\int_{-a}^{2a}\frac{x}{1+x^2}\,dx=\lim_{a\to\infty}\left[\frac{1}{2}\ln\left(1+x^2\right)\right]_{-a}^{2a}$$

$$=\lim_{a\to\infty}\left(\frac{1}{2}\ln\left(1+(2a)^2\right)-\frac{1}{2}\ln\left(1+(-a)^2\right)\right)$$

$$=\lim_{a\to\infty}\frac{1}{2}\ln\frac{1+4a^2}{1+a^2}=\frac{1}{2}\ln 4=\ln 2$$

$$\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,dx=\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx$$

$$\lim_{a\to\infty}\int_{-a}^{2a}\frac{x}{1+x^2}\,dx=\int_{-\infty}^{\infty}\frac{x}{1+x^2}\,dx$$

but $\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,dx\neq\lim_{a\to\infty}\int_{-a}^{2a}\frac{x}{1+x^2}\,dx$

so mean does not exist