Volumes of revolution 4D

1 a $y = 0 \Rightarrow a = \dfrac{2000}{20} = 100$

Diameter $= 2a = 200$ metres.

b Volume $= \pi\int_0^{120}\left(\dfrac{2000}{20+y}\right)^2 \mathrm{d}y$

$= 4\,000\,000\pi\int_0^{120}\dfrac{1}{(20+y)^2}\mathrm{d}y$

$= 4\,000\,000\pi\left[-\dfrac{1}{20+y}\right]_0^{120}$

$= 4\,000\,000\pi\left(-\dfrac{1}{140}+\dfrac{1}{20}\right)$

$= 4\,000\,000\pi\times\dfrac{6}{140}$

$= \dfrac{1\,200\,000\pi}{7}$ m^3

2 Volume of water $= \pi\int_0^{20}\dfrac{100y}{10y^2+1}\mathrm{d}y$

$= \pi\left[5\ln\left(10y^2+1\right)\right]_0^{20}$

$= 5\pi\ln 4001\,\text{cm}^3$

So $p = 5, q = 4001$

3 a $\cos 3\theta = \cos(2\theta+\theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= \left(2\cos^2\theta - 1\right)\cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\left(1-\cos^2\theta\right)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$

$4\cos^3\theta = 3\cos\theta + \cos 3\theta$

$\cos^3\theta = \dfrac{3}{4}\cos\theta + \dfrac{1}{4}\cos 3\theta$

b $x = 50\cos\theta \Rightarrow x^2 = 2500\cos^2\theta$

$y = 30\sin\theta \Rightarrow \dfrac{dy}{d\theta} = 30\cos\theta$

3 b Volume of tent

$= \pi\int_0^{\frac{\pi}{2}}\left(2500\cos^2\theta\times 30\cos\theta\right)\mathrm{d}\theta$

$= 75\,000\pi\int_0^{\frac{\pi}{2}}\cos^3\theta\,\mathrm{d}\theta$

$= \dfrac{75\,000}{4}\pi\int_0^{\frac{\pi}{2}}\left(3\cos\theta + \cos 3\theta\right)\mathrm{d}\theta$

$= \dfrac{75\,000}{4}\pi\left[3\sin\theta + \dfrac{1}{3}\sin 3\theta\right]_0^{\frac{\pi}{2}}$

$= \dfrac{75\,000}{4}\pi\left(3-\dfrac{1}{3}\right)$

$= 50\,000\pi$ m^3

4 a $x = \sin y\sqrt{\sin 2y} \Rightarrow x^2 = 2\sin^3 y\cos y$

Volume $= 2\pi\int_0^{\frac{\pi}{2}}\sin^3 y\cos y\,\mathrm{d}y$

$= 2\pi\left[\dfrac{1}{4}\sin^4 y\right]_0^{\frac{\pi}{2}}$

$= \dfrac{\pi}{2}$ m^3

b Linear scale factor $= 6\pi\Big/\dfrac{\pi}{2} = 12$

Volume scale factor $= 12^3 = 1728$

Volume of real hot-air balloon $= 1728\times\dfrac{\pi}{2}$

$= 864\pi$ m^3

5 a $x = 2\sin 2\theta \quad y = 3\sin\theta$

$x^2 = 4\sin^2 2\theta = 16\sin^2\theta\cos^2\theta$

$= 16\sin^2\theta(1-\sin^2\theta)$

$= 16\sin^2\theta - 16\sin^4\theta$

$= 16\left(\dfrac{y}{3}\right)^2 - 16\left(\dfrac{y}{3}\right)^4$

$= 16\dfrac{y^2}{9} - 16\dfrac{y^4}{81}$

$= \dfrac{16}{81}y^2\left(9-y^2\right)$

5 b For the model, when $x = 0,\ y = 3$

$$\text{Volume } = \frac{16}{81}\pi\int_0^3 \left(9y^2 - y^4\right)\text{d}y$$
$$= \frac{16}{81}\pi\left[3y^3 - \frac{y^5}{5}\right]_0^3$$
$$= \frac{16}{81}\pi\left(81 - \frac{243}{5}\right)$$
$$= \frac{16}{81}\pi \times \frac{162}{5}$$
$$= \frac{32}{5}\pi\ \text{mm}^3$$

Maximum number of earrings
= integer value of $300\Big/\frac{32}{5}\pi = 14$

c i e.g. patterned earring may mean that earring requires less material.

ii e.g. wasted material upon transfer to mould.