Volumes of revolution 4C

1 $y = t^2 \Rightarrow y^2 = t^4$

$x = t^3 \Rightarrow \frac{dx}{dt} = 3t^2$

$x = 0 \Rightarrow t = 0$

$x = 8 \Rightarrow t = 2$

$\text{Volume} = \pi\int_0^2 y^2 \frac{dx}{dt}\,dt$

$= \pi\int_0^2 3t^6\,dt$

$= \pi\left[\frac{3t^7}{7}\right]_0^2$

$= \frac{384\pi}{7}$

2 a $x = e^t$

$x = e^2 \Rightarrow t = 2$

$x = e^3 \Rightarrow t = 3$

b $x = e^t \Rightarrow \frac{dx}{dt} = e^t$

$y = \sqrt{t-1} \Rightarrow y^2 = t-1$

$\text{Volume} = \pi\int_2^3 (t-1)e^t\,dt$

$= \pi\int_2^3 (te^t - e^t)\,dt$

$= \pi\left(\int_2^3 te^t\,dt - \int_2^3 e^t\,dt\right)$

$= \pi\left(\int_2^3 te^t\,dt - e^3 + e^2\right)$

Let $u = t$ and $\frac{dv}{dt} = e^t$

So $\frac{du}{dt} = 1$ and $v = e^t$

Integrating by parts

$\text{Volume} = \pi\left(\left[te^t\right]_2^3 - \int_2^3 e^t\,dt - e^3 + e^2\right)$

$= \pi\left(\left(3e^3 - 2e^2\right) - \left[e^t\right]_2^3 - e^3 + e^2\right)$

$= \pi\left(\left(3e^3 - 2e^2\right) - \left(e^3 - e^2\right) - e^3 + e^2\right)$

$= \pi e^3$

2 c $y^2 = t - 1 \Rightarrow t = y^2 + 1$

$x = e^t \Rightarrow t = \ln x$

$y^2 + 1 = \ln x$

$y^2 = \ln x - 1$

d $I = \pi\int_{e^2}^{e^3} (\ln x - 1)\,dx$

$= \pi\left(\int_{e^2}^{e^3} \ln x\,dx - \int_{e^2}^{e^3} dx\right)$

$= \pi\left(\int_{e^2}^{e^3} \ln x\,dx - e^3 + e^2\right)$

Let $u = \ln x$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

Integrating by parts

$I = \pi\left(\left[x\ln x\right]_{e^2}^{e^3} - \int_{e^2}^{e^3} dx - e^3 + e^2\right)$

$= \pi\left(3e^3 - 2e^2 - e^3 + e^2 - e^3 + e^2\right)$

$= \pi e^3$

3 a $x = \sqrt{1 - \sin\theta} \Rightarrow x^2 = 1 - \sin\theta$

$\sin\theta = 1 - x^2$

$\sin^2\theta = 1 - 2x^2 + x^4$

$y = \cos\theta \Rightarrow \cos^2\theta = y^2$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow 1 - 2x^2 + x^4 + y^2 = 1$

$y^2 = 2x^2 - x^4$

b $y = 0 \Rightarrow 2x^2 - x^4 = 0$

$x^2(2 - x^2) = 0$

$x = 0$ or $\sqrt{2}$

P has coordinates $\left(\sqrt{2}, 0\right)$

3 c Volume $= \pi\int_0^{\sqrt{2}}\left(2x^2 - x^4\right)\mathrm{d}x$

$$= \pi\left[\frac{2x^3}{3} - \frac{x^5}{5}\right]_0^{\sqrt{2}}$$

$$= \pi\left(\left(\frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5}\right) - 0\right)$$

$$= \frac{8\sqrt{2}}{15}\pi$$

4 a $x = \tan\theta,\ y = \sec^3\theta$

$y = 1 \Rightarrow \sec\theta = 1 \Rightarrow \theta = 0$

$y = 8 \Rightarrow \sec\theta = 2 \Rightarrow \theta = \frac{\pi}{3}$

b $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 3\sec^3\theta\tan\theta$

Volume $= \pi\int_0^{\frac{\pi}{3}} x^2\frac{\mathrm{d}y}{\mathrm{d}\theta}\mathrm{d}\theta$

$$= 3\pi\int_0^{\frac{\pi}{3}}\sec^3\theta\tan^3\theta\,\mathrm{d}\theta$$

$$= 3\pi\int_0^{\frac{\pi}{3}}\sec^3\theta\tan\theta\left(\sec^2\theta - 1\right)\mathrm{d}\theta$$

$$= 3\pi\int_0^{\frac{\pi}{3}}\left(\sec^5\theta\tan\theta - \sec^3\theta\tan\theta\right)\mathrm{d}\theta$$

$$= 3\pi\left[\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right]_0^{\frac{\pi}{3}}$$

$$= 3\pi\left(\left(\frac{32}{5} - \frac{8}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right)$$

$$= 3\pi\left(\frac{56}{15} + \frac{2}{15}\right)$$

$$= \frac{58\pi}{5}$$

c $\tan^2\theta = \sec^2\theta - 1$

So $x^2 = y^{\frac{2}{3}} - 1$

4 d $\pi\int_1^8 x^2\,\mathrm{d}y = \pi\int_1^8\left(y^{\frac{2}{3}} - 1\right)\mathrm{d}y$

$$= \pi\left[\frac{3}{5}y^{\frac{5}{3}} - y\right]_1^8$$

$$= \pi\left(\left(\frac{96}{5} - 8\right) - \left(\frac{3}{5} - 1\right)\right)$$

$$= \pi\left(\frac{56}{5} + \frac{2}{5}\right)$$

$$= \frac{58\pi}{5}$$

5 a $x = \sin^4\theta\sqrt{\cos\theta}$

$y = \cos\theta \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = -\sin\theta$

Lower bound for y is when $\theta = \frac{\pi}{2}$

Upper bound for y is when $\theta = 0$

Volume $= \pi\int_{\frac{\pi}{2}}^{0} -\sin^9\theta\cos\theta\,\mathrm{d}\theta$

$$= \frac{\pi}{10}\left[-\sin^{10}\theta\right]_{\frac{\pi}{2}}^{0}$$

$$= \frac{\pi}{10}$$

6 $y = t^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2t$

$t = \pm 2 \Rightarrow a = 4$

Required volume is that generated by the curve for $0 \le t \le 2$.

Volume $= \pi\int_0^2 x^2\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$

$$= \pi\int_0^2 8t^3\,\mathrm{d}t$$

$$= \pi\left[2t^4\right]_0^2$$

$$= 32\pi$$

7 a $\int \cos^2\theta \, d\theta = \frac{1}{2}\int(\cos 2\theta + 1)\, d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$

$= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + c$

b $x = \cot\theta \Rightarrow \frac{dx}{d\theta} = -\mathrm{cosec}^2\theta$

$x = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$

$x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$

Volume $= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} 16\sin^2 2\theta(-\mathrm{cosec}^2\theta)\, d\theta$

$= -\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{16\sin^2 2\theta}{\sin^2\theta}\, d\theta$

$= -16\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{4\sin^2\theta\cos^2\theta}{\sin^2\theta}\, d\theta$

$= -64\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \cos^2\theta\, d\theta$

$k = -64,\ a = \frac{\pi}{3},\ b = \frac{\pi}{6}$

c Volume $= 64\pi\left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= 64\pi\left(\left(\frac{\sqrt{3}}{8} + \frac{\pi}{6}\right) - \left(\frac{\sqrt{3}}{8} + \frac{\pi}{12}\right)\right)$

$= 64\pi\left(\frac{\pi}{6} - \frac{\pi}{12}\right)$

$= \frac{64}{12}\pi^2$

$= \frac{16}{3}\pi^2$

8 $y = \ln 2t \Rightarrow t = \frac{1}{2}e^y$

$x = \frac{1}{2t} = e^{-y}$

Volume $= \pi\int_0^a e^{-2y}\, dy$

$= \pi\left[-\frac{1}{2}e^{-2y}\right]_0^a$

$= \pi\left(-\frac{1}{2}e^{-2a} + \frac{1}{2}\right)$

$= \frac{\pi}{2}(1 - e^{-2a}) = \frac{24\pi}{49}$

$1 - e^{-2a} = \frac{48}{49}$

$e^{-2a} = \frac{1}{49}$

$e^{2a} = 49$

$2a = \ln 49$

$a = \frac{1}{2}\ln 49 = \ln 7$

9 $y = t^2 \Rightarrow \frac{dy}{dt} = 2t$

Volume $= \pi\int_0^\pi 8t\sin^2 t\, dt$

$= 4\pi\int_0^\pi t(1 - \cos 2t)\, dt$

$= 4\pi\left(\left[\frac{t^2}{2}\right]_0^\pi - \int_0^\pi t\cos 2t\, dt\right)$

$= 4\pi\left(\frac{\pi^2}{2} - \int_0^\pi t\cos 2t\, dt\right)$

Let $u = t$ and $\frac{dv}{dt} = \cos 2t$

So $\frac{du}{dt} = 1$ and $v = \frac{1}{2}\sin 2t$

Integrating by parts

Volume $= 4\pi\left(\frac{\pi^2}{2} - \left[\frac{t}{2}\sin 2t\right]_0^\pi + \frac{1}{2}\int_0^\pi \sin 2t\, dt\right)$

$= 4\pi\left(\frac{\pi^2}{2} + \frac{1}{2}\int_0^\pi \sin 2t\, dt\right)$

$= 4\pi\left(\frac{\pi^2}{2} + \frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_0^\pi\right)$

$= 2\pi^3$

10 $x = t^2 - 2t \Rightarrow \dfrac{dx}{dt} = 2t - 2$

Volume generated by curve

$$= \pi\int_1^{-1}\left(1-t^2\right)^2\left(2t-2\right)dt$$

$$= 2\pi\int_1^{-1}\left(1-2t^2+t^4\right)\left(t-1\right)dt$$

$$= 2\pi\int_1^{-1}\left(t-2t^3+t^5-1+2t^2-t^4\right)dt$$

$$= 2\pi\int_1^{-1}\left(t^5-t^4-2t^3+2t^2+t-1\right)dt$$

$$= 2\pi\left[\frac{t^6}{6}-\frac{t^5}{5}-\frac{t^4}{2}+\frac{2t^3}{3}+\frac{t^2}{2}-t\right]_1^{-1}$$

$$= 2\pi\left(\left(\frac{1}{6}+\frac{1}{5}-\frac{1}{2}-\frac{2}{3}+\frac{1}{2}+1\right)-\left(\frac{1}{6}-\frac{1}{5}-\frac{1}{2}+\frac{2}{3}+\frac{1}{2}-1\right)\right)$$

$$= \frac{32}{15}\pi$$

P coordinates are $(-1,0)$

Q coordinates are $(0,1)$

S coordinates are $(3,0)$

Volume of cone generated by line PQ

$$= \frac{1}{3}\pi\times 1^2\times 1 = \frac{\pi}{3}$$

Volume of cone generated by line QS

$$= \frac{1}{3}\pi\times 1^2\times 3 = \pi$$

Volume generated by R

$$= \left(\frac{32}{15}-\frac{1}{3}-1\right)\pi$$

$$= \frac{12}{15}\pi$$

$$= \frac{4}{5}\pi$$

11 a $x = e^t \quad y = e^{-2t}$

$$\frac{dy}{dt} = -2e^{-2t}$$

$y = 1 \Rightarrow t = 0$

$y = 6 \Rightarrow t = -\frac{1}{2}\ln 6$

$$\text{Volume } = \pi\int_0^{-\frac{1}{2}\ln 6} x^2\frac{dy}{dt}dt$$

$$= \pi\int_0^{-\frac{1}{2}\ln 6} e^{2t}\left(-2e^{-2t}\right)dt$$

$$= -2\pi\int_0^{-\frac{1}{2}\ln 6} dt$$

$$= -2\pi\left[t\right]_0^{-\frac{1}{2}\ln 6}$$

$$= -2\pi\left(-\frac{1}{2}\ln 6 - 0\right)$$

$$= \pi\ln 6$$

b Find the gradient of the tangent at (1, 1).

$$\frac{dy}{dt} = -2e^{-2t} \text{ and } \frac{dx}{dt} = e^t$$

$$\frac{dy}{dx} = -\frac{2e^{-2t}}{e^t} = -2e^{-3t}$$

When $y = 1$, $t = 0$ and $\dfrac{dy}{dx} = -2$

Equation of tangent is

$y - 1 = -2(x-1)$

$y = -2x + 3$

So the tangent cuts the y axis at (0, 3).

Volume generated by S is the volume generated by R minus the volume of the cone of base 1 and height 2.

$$\text{Volume of cone } = \frac{1}{3}\pi\times 1^2\times 2 = \frac{2}{3}\pi$$

Volume generated by S

$$= \pi\ln 6 - \frac{2}{3}\pi$$

$$= \pi\left(\ln 6 - \frac{2}{3}\right)$$