Volumes of revolution 4B

1 a Volume $= \pi\int_0^1 x^2\ \mathrm{d}y$

$= \pi\int_0^1 \left(e^{2y} - e^{-y}\right)^2\ \mathrm{d}y$

$= \pi\int_0^1 \left(e^{4y} - 2e^{y} + e^{-2y}\right)\mathrm{d}y$

$= \pi\left[\frac{1}{4}e^{4y} - 2e^{y} - \frac{1}{2}e^{-2y}\right]_0^1$

$= \pi\left(\left(\frac{1}{4}e^{4} - 2e - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{4} - 2 - \frac{1}{2}\right)\right)$

$= \pi\left(\frac{1}{4}e^{4} - 2e - \frac{1}{2}e^{-2} + \frac{9}{4}\right)$

b Volume $= \pi\int_0^1 x^2\ \mathrm{d}y$

$= \pi\int_0^1 ye^{2y^2}\ \mathrm{d}y$

$= \pi\left[\frac{1}{4}e^{2y^2}\right]_0^1$

$= \pi\left(\frac{1}{4}e^{2} - \frac{1}{4}\right)$

$= \frac{\pi}{4}\left(e^2 - 1\right)$

c Volume $= \pi\int_1^5 x^2\ \mathrm{d}y$

$= \pi\int_1^5 \left(\frac{5 - \ln y}{y^2}\right)\mathrm{d}y$

$= \pi\int_1^5 \left(\frac{5}{y^2} - \frac{\ln y}{y^2}\right)\mathrm{d}y$

$= \pi\left(\left[-\frac{5}{y}\right]_1^5 - \int_1^5 \frac{\ln y}{y^2}\ \mathrm{d}y\right)$

$= \pi\left(\left(-1 - (-5)\right) - \int_1^5 \frac{\ln y}{y^2}\ \mathrm{d}y\right)$

$= \pi\left(4 - \int_1^5 \frac{\ln y}{y^2}\ \mathrm{d}y\right)$

Let $u = \ln y$ and $\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{1}{y^2}$

So $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{y}$ and $v = -\frac{1}{y}$

1 c Integrating by parts

Volume $= \pi\left(4 - \left(\left[-\frac{\ln y}{y}\right]_1^5 + \int_1^5 \frac{1}{y^2}\ \mathrm{d}y\right)\right)$

$= \pi\left(4 + \frac{\ln 5}{5} - \left[-\frac{1}{y}\right]_1^5\right)$

$= \pi\left(4 + \frac{\ln 5}{5} - \left(-\frac{1}{5} + 1\right)\right)$

$= \frac{\pi}{5}\left(\ln 5 + 16\right)$

d Volume $= \pi\int_0^1 x^2\ \mathrm{d}y$

$= \pi\int_{e^4}^{e^9} \frac{1}{y\ln y}\ \mathrm{d}y$

$= \pi\left[\ln\left(\ln y\right)\right]_{e^4}^{e^9}$

$= \pi\left(\ln 9 - \ln 4\right)$

$= \pi\left(\ln\frac{9}{4}\right)$

2 a $y = \frac{1}{x} - 1 \Rightarrow x = \frac{1}{y+1}$

Volume $= \pi\int_0^1 \frac{1}{(y+1)^2}\ \mathrm{d}y$

$= \pi\left[-\frac{1}{y+1}\right]_0^1$

$= \pi\left(-\frac{1}{2} + 1\right)$

$= \frac{\pi}{2}$

b $y = \frac{5 - 2x^2}{x^2 - 1} \Rightarrow x^2 = \frac{y+5}{y+2}$

Volume $= \pi\int_{-1}^1 \frac{y+5}{y+2}\ \mathrm{d}y$

$= \pi\int_{-1}^1 \left(1 + \frac{3}{y+2}\right)\mathrm{d}y$

$= \pi\left[y + 3\ln\left(y+2\right)\right]_{-1}^1$

$= \pi\left(\left(1 + 3\ln 3\right) - (-1)\right)$

$= \pi\left(2 + 3\ln 3\right)$

2 c $y = 2e^{x^2} \Rightarrow x^2 = \ln\left(\frac{y}{2}\right)$

$$\text{Volume } = \pi\int_2^4 \ln\left(\frac{y}{2}\right) dy$$
$$= \pi\int_2^4 (\ln y - \ln 2)\, dy$$

Let $u = \ln y$ and $\frac{dv}{dy} = 1$

So $\frac{du}{dy} = \frac{1}{y}$ and $v = y$

Integrating by parts

$$\text{Volume } = \pi\left([y\ln y]_2^4 - \int_2^4 dy - [y\ln 2]_2^4\right)$$
$$= \pi\left((4\ln 4 - 2\ln 2) - [y]_2^4 - (4\ln 2 - 2\ln 2)\right)$$
$$= \pi(6\ln 2 - 2 - 2\ln 2)$$
$$= \pi(4\ln 2 - 2)$$

d $y = \arccos\sqrt{x} \Rightarrow x^2 = \cos^4 y$

$$\text{Volume } = \pi\int_0^{\frac{\pi}{2}} \cos^4 y\, dy$$

$$\cos^4 y = (\cos^2 y)^2$$
$$= \left(\frac{1}{2}(\cos 2y + 1)\right)^2$$
$$= \frac{1}{4}(\cos^2 2y + 2\cos 2y + 1)$$
$$= \frac{1}{4}\left(\frac{1}{2}(\cos 4y + 1) + 2\cos 2y + 1\right)$$
$$= \frac{1}{8}(\cos 4y + 4\cos 2y + 3)$$

$$\text{Volume } = \frac{\pi}{8}\int_0^{\frac{\pi}{2}} (\cos 4y + 4\cos 2y + 3)\, dy$$
$$= \frac{\pi}{8}\left[\frac{1}{4}\sin 4y + 2\sin 2y + 3y\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{8}\left(\frac{3\pi}{2}\right)$$
$$= \frac{3\pi^2}{16}$$

3 $$\text{Volume } = \pi\int_1^b \frac{1}{(2y+1)^2}\, dy$$
$$= \pi\left[-\frac{1}{2}\left(\frac{1}{2y+1}\right)\right]_1^b$$
$$= -\frac{\pi}{2}\left(\frac{1}{2b+1} - \frac{1}{3}\right)$$
$$= \frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{2b+1}\right)$$

So $\frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{2b+1}\right) = \frac{\pi}{10}$

$$\frac{1}{3} - \frac{1}{2b+1} = \frac{1}{5}$$
$$\frac{1}{2b+1} = \frac{2}{15}$$
$$2b + 1 = \frac{15}{2}$$
$$2b = \frac{13}{2}$$
$$b = \frac{13}{4}$$

4 a $x = \sqrt{y}\sin y$

When $x = 0$, $\sin y = 0 \Rightarrow y = 0, \pi, 2\pi, \ldots$

So $b = \pi$

4 b Volume $= \pi\int_0^{\pi} x^2 \, dy$

$$= \pi\int_0^{\pi} y\sin^2 y \, dy$$

$$= \frac{\pi}{2}\int_0^{\pi} y(1-\cos 2y)\,dy$$

$$= \frac{\pi}{2}\left(\left[\frac{y^2}{2}\right]_0^{\pi} - \int_0^{\pi} y\cos 2y \, dy\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi^2}{2} - \int_0^{\pi} y\cos 2y \, dy\right)$$

Let $u = y$ and $\frac{dv}{dy} = \cos 2y$

So $\frac{du}{dy} = 1$ and $v = \frac{1}{2}\sin 2y$

Integrating by parts

Volume

$$= \frac{\pi}{2}\left(\frac{\pi^2}{2} - \left(\left[\frac{y}{2}\sin 2y\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin 2y \, dy\right)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi^2}{2} - \frac{1}{4}[\cos 2y]_0^{\pi}\right)$$

$$= \frac{\pi^3}{4}$$

5 $y = 3\ln(x-1)$

$$\frac{y}{3} = \ln(x-1)$$

$$x - 1 = e^{\frac{y}{3}}$$

$$x = e^{\frac{y}{3}} + 1$$

Volume $= \pi\int_0^5 x^2 \, dy$

$$= \pi\int_0^5 \left(e^{\frac{y}{3}} + 1\right)^2 dy$$

$$= \pi\int_0^5 \left(e^{\frac{2y}{3}} + 2e^{\frac{y}{3}} + 1\right) dy$$

$$= \pi\left[\frac{3}{2}e^{\frac{2y}{3}} + 6e^{\frac{y}{3}} + y\right]_0^5$$

$$= \pi\left(\left(\frac{3}{2}e^{\frac{10}{3}} + 6e^{\frac{5}{3}} + 5\right) - \left(\frac{3}{2} + 6\right)\right)$$

$$= \pi\left(\frac{3}{2}e^{\frac{10}{3}} + 6e^{\frac{5}{3}} - \frac{5}{2}\right)$$

6 a $\cos y + \sqrt{3}\sin y = 2\left(\frac{1}{2}\cos y + \frac{\sqrt{3}}{2}\sin y\right)$

$$= 2(\cos y\cos\alpha + \sin y\sin\alpha) \text{ where } \alpha = \frac{\pi}{3}$$

$$= 2\cos\left(y - \frac{\pi}{3}\right)$$

b Volume $= \pi\int_0^{\frac{\pi}{3}} x^2 \, dy$

$$= \pi\int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos y + \sqrt{3}\sin y}\right)^2 dy$$

$$= \pi\int_0^{\frac{\pi}{3}} \frac{1}{4\cos^2\left(y - \frac{\pi}{3}\right)} dy$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{3}} \sec^2\left(y - \frac{\pi}{3}\right) dy$$

$$= \frac{\pi}{4}\left[\tan\left(y - \frac{\pi}{3}\right)\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{4}\left(\tan 0 - \tan\left(-\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi\sqrt{3}}{4}$$

7 a $I = \int_0^1 \frac{2^y}{(2^y+1)^2}\,dy$

Let $u = 2^y \Rightarrow \frac{du}{dy} = 2^y \ln 2$

$y = 1 \Rightarrow u = 2$

$y = 0 \Rightarrow u = 1$

$$I = \int_1^2 \frac{u}{(u+1)^2 u \ln 2}\,du$$
$$= \frac{1}{\ln 2}\int_1^2 \frac{1}{(u+1)^2}\,du$$
$$= \frac{1}{\ln 2}\left[-\frac{1}{u+1}\right]_1^2$$
$$= \frac{1}{\ln 2}\left(-\frac{1}{3}+\frac{1}{2}\right)$$
$$= \frac{1}{6\ln 2}$$

b Volume $= \pi\int_0^1 x^2\,dy$

$$= \pi\int_0^1 \frac{2^y}{(2^y+1)^2}\,dy$$
$$= \pi \times \frac{1}{6\ln 2}$$
$$= \frac{\pi}{6\ln 2}$$

8 a By De Moivre's Theorem

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$
$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

and

$$\sin 5\theta = 5\cos^4\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$
$$= 5\sin\theta - 5(3\sin\theta - \sin 3\theta) + 16\sin^5\theta$$
$$= -10\sin\theta + 5\sin 3\theta + 16\sin^5\theta$$
$$\sin^5\theta = \frac{1}{16}(10\sin\theta - 5\sin 3\theta + \sin 5\theta)$$

8 b $x = \sin^2 y \sin\sqrt{y} \Rightarrow x^2 = \sin^5 y$

Volume $= \pi\int_{\frac{\pi}{4}}^{\pi} \sin^5 y\,dy$

$$= \frac{\pi}{16}\int_{\frac{\pi}{4}}^{\pi}(10\sin y - 5\sin 3y + \sin 5y)\,dy$$
$$= \frac{\pi}{16}\left[-10\cos y + \frac{5}{3}\cos 3y - \frac{1}{5}\cos 5y\right]_{\frac{\pi}{4}}^{\pi}$$
$$= \frac{\pi}{16}\left(\left(10 - \frac{5}{3} + \frac{1}{5}\right) - \left(-\frac{10}{\sqrt{2}} - \frac{5}{3\sqrt{2}} + \frac{1}{5\sqrt{2}}\right)\right)$$
$$= \frac{\pi}{16}\left(\frac{150-25+3}{15} + \frac{150+25-3}{15\sqrt{2}}\right)$$
$$= \frac{\pi}{16}\left(\frac{128}{15} + \frac{172}{15\sqrt{2}}\right)$$
$$= \frac{\pi}{16}\left(\frac{128}{15} + \frac{86\sqrt{2}}{15}\right)$$
$$= \frac{\pi}{15}\left(8 + \frac{43\sqrt{2}}{8}\right)$$