Volumes of revolution 4A

1 a Volume $= \pi\int_0^2 \left(\frac{2}{x+1}\right)^2 dx$

$= \pi\int_0^2 \frac{4}{(x+1)^2} dx$

$= \pi\left[-\frac{4}{x+1}\right]_0^2$

$= \pi\left(\left(-\frac{4}{3}\right)-(-4)\right)$

$= \frac{8\pi}{3}$

b Volume $= \pi\int_0^{\frac{\pi}{2}} \left(\frac{4\sin x}{1+\cos x}\right) dx$

Let $u = 1+\cos x \Rightarrow \frac{du}{dx} = -\sin x$

Volume $= \pi\int_2^1 \left(-\frac{4}{u}\right) du$

$= \pi\int_1^2 \frac{4}{u} du$

$= \pi[4\ln u]_1^2$

$= 4\pi\ln 2$

c Volume $= \pi\int_0^{\frac{\pi}{4}} \left(\sqrt{x}\sec x\right)^2 dx$

$= \pi\int_0^{\frac{\pi}{4}} x\sec^2 x\, dx$

Let $u = x$ and $\frac{dv}{dx} = \sec^2 x$

So $\frac{du}{dx} = 1$ and $v = \tan x$

Integrating by parts

Volume $= \pi\left([x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\, dx\right)$

$= \pi\left(\left(\frac{\pi}{4}-0\right)-\left[\ln(\sec x)\right]_0^{\frac{\pi}{4}}\right)$

$= \pi\left(\frac{\pi}{4}-\left(\ln\sqrt{2}-\ln 1\right)\right)$

$= \pi\left(\frac{\pi}{4}-\ln\sqrt{2}\right)$

$= \frac{\pi}{4}(\pi-\ln 4)$

1 d Volume $= \pi\int_0^2 \frac{5x}{10x^2+1} dx$

$= \pi\left[\frac{1}{4}\ln\left(10x^2+1\right)\right]_0^2$

$= \frac{\pi}{4}(\ln 41-\ln 1)$

$= \frac{\pi}{4}\ln 41$

e Volume $= \pi\int_1^2 \frac{\ln x}{x^2} dx$

Let $u = \ln x$ and $\frac{dv}{dx} = \frac{1}{x^2}$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = -\frac{1}{x}$

Integrating by parts

Volume $= \pi\left(\left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 \frac{1}{x^2} dx\right)$

$= \pi\left(-\frac{\ln 2}{2}+\left[-\frac{1}{x}\right]_1^2\right)$

$= \pi\left(-\frac{\ln 2}{2}+\left(-\frac{1}{2}+1\right)\right)$

$= \frac{\pi}{2}(1-\ln 2)$

f Volume $= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\operatorname{cosec} x+\cot x)^2 dx$

$= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\operatorname{cosec}^2 x+2\operatorname{cosec} x\cot x+\cot^2 x\right) dx$

$= \pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(2\operatorname{cosec}^2 x+2\operatorname{cosec} x\cot x-1\right) dx$

$= \pi[-2\cot x-2\operatorname{cosec} x-x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \pi\left(\left(-2-\frac{\pi}{2}\right)-\left(-\frac{2}{\sqrt{3}}-\frac{4}{\sqrt{3}}-\frac{\pi}{3}\right)\right)$

$= \pi\left(2\sqrt{3}-2-\frac{\pi}{6}\right)$

2 Volume $= \pi\int_0^{\frac{\pi}{2}} \cos^2 x \sin 2x \, dx$

$= \pi\int_0^{\frac{\pi}{2}} 2\cos^3 x \sin x \, dx$

$= \pi\left[-\frac{1}{2}\cos^4 x\right]_0^{\frac{\pi}{2}}$

$= \pi\left(0-\left(-\frac{1}{2}\right)\right)$

$= \frac{\pi}{2}$

3 $y = \ln x$

When $y = 0$, $\ln a = 0 \Rightarrow a = 1$

Volume $= \pi\int_1^3 (\ln x)^2 \, dx$

Let $u = (\ln x)^2$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{2}{x}\ln x$ and $v = x$

Integrating by parts

Volume $= \pi\left(\left[x(\ln x)^2\right]_1^3 - 2\int_1^3 \ln x \, dx\right)$

$= \pi\left(3(\ln 3)^2 - 2\int_1^3 \ln x \, dx\right)$

Now let $u = \ln x$ and $\frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{x}$ and $v = x$

Integrating by parts again, Volume

$= \pi\left(3(\ln 3)^2 - 2[x\ln x]_1^3 + 2\int_1^3 dx\right)$

$= \pi\left(3(\ln 3)^2 - 6\ln 3 + 2[x]_1^3\right)$

$= \pi\left(3(\ln 3)^2 - 6\ln 3 + 4\right)$

4 a $I = \int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} \frac{1}{x^2\sqrt{9-x^2}} \, dx$

Let $x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

When $x = \frac{3\sqrt{2}}{2}$, $\theta = \frac{\pi}{4}$

When $x = \frac{3}{2}$, $\theta = \frac{\pi}{6}$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{3\cos\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} \, d\theta$

$= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{3\cos\theta}{3\sin^2\theta\sqrt{1-\sin^2\theta}} \, d\theta$

$= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin^2\theta} \, d\theta$

$= \frac{1}{9}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec}^2\theta \, d\theta$

$= \frac{1}{9}[-\cot\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{9}\left(-1-\left(-\sqrt{3}\right)\right)$

$= \frac{1}{9}\left(\sqrt{3}-1\right)$

b Volume $= \pi\int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} y^2 \, dx$

$= \pi\int_{\frac{3}{2}}^{\frac{3\sqrt{2}}{2}} \frac{81}{x^2\sqrt{9-x^2}} \, dx$

$= 81\pi \times I$ from part *a* above

$= 81\pi \times \frac{1}{9}\left(\sqrt{3}-1\right)$

$= 9\pi\left(\sqrt{3}-1\right)$

5 $y^2 = \frac{4x+3}{(x+2)(2x-1)}$

$\frac{4x+3}{(x+2)(2x-1)} = \frac{A}{x+2} + \frac{B}{2x-1}$

$4x+3 = A(2x-1) + B(x+2)$

$x = \frac{1}{2} \Rightarrow 5 = \frac{5B}{2} \Rightarrow B = 2$

$x = -2 \Rightarrow -5 = -5A \Rightarrow A = 1$

So $y^2 = \frac{1}{x+2} + \frac{2}{2x-1}$

5 Volume $= \pi\int_1^4 y^2 \, dx$

$= \pi\int_1^4 \left(\frac{1}{x+2} + \frac{2}{2x-1}\right) dx$

$= \pi\left[\ln(x+2) + \ln(2x-1)\right]_1^4$

$= \pi(\ln 6 + \ln 7 - \ln 3 - \ln 1)$

$= \pi \ln 14$

6 a $2y^2 = x\sin x + x$

$y = 0 \Rightarrow x\sin x + x = 0$

So $x(\sin x + 1) = 0$

$x = 0$ or $\sin x = -1$

$x = 0$ or $x = \frac{3\pi}{2}$

Coordinates of point A are $\left(\frac{3\pi}{2}, 0\right)$

b Volume $= \pi\int_0^{\frac{3\pi}{2}} y^2 \, dx$

$= \frac{\pi}{2}\int_0^{\frac{3\pi}{2}} (x\sin x + x) \, dx$

$= \frac{\pi}{2}\left(\int_0^{\frac{3\pi}{2}} x\sin x \, dx + \int_0^{\frac{3\pi}{2}} x \, dx\right)$

Let $u = x$ and $\frac{dv}{dx} = \sin x$

So $\frac{du}{dx} = 1$ and $v = -\cos x$

Volume

$= \frac{\pi}{2}\left([-x\cos x]_0^{\frac{3\pi}{2}} + \int_0^{\frac{3\pi}{2}} \cos x \, dx + \left[\frac{x^2}{2}\right]_0^{\frac{3\pi}{2}}\right)$

$= \frac{\pi}{2}\left(0 + [\sin x]_0^{\frac{3\pi}{2}} + \frac{9\pi^2}{8}\right)$

$= \frac{\pi}{2}\left(\frac{9\pi^2}{8} - 1\right)$

7 a Volume $= \pi\int_{-1}^2 \left(\frac{10}{3(5+2x)}\right)^2 dx$

$= \frac{100\pi}{9}\int_{-1}^2 \frac{1}{(5+2x)^2} \, dx$

$= -\frac{100\pi}{18}\left[\frac{1}{5+2x}\right]_{-1}^2$

$= -\frac{100\pi}{18}\left(\frac{1}{9} - \frac{1}{3}\right)$

$= \frac{100\pi}{18} \times \frac{2}{9}$

$= \frac{100\pi}{81}$

b Volume generated by $y = \frac{20}{3(5+2x)}$

$= \frac{400\pi}{9}\int_{-1}^2 \frac{1}{(5+2x)^2} \, dx$

$= 4 \times \frac{100\pi}{81}$ (answer to part a)

Volume generated by the area between the two curves will be

$= \frac{400\pi}{81} - \frac{100\pi}{81}$

$= \frac{300\pi}{81}$

$= \frac{100\pi}{27}$

8 $y = xe^{-x}$ crosses the line $y = \frac{1}{4}x$

When $xe^{-x} = \frac{1}{4}x$

$xe^{-x} - \frac{1}{4}x = 0$

$x\left(e^{-x} - \frac{1}{4}\right) = 0$

$x = 0$ or $e^{-x} = \frac{1}{4}$

$x = 0$ or $x = \ln 4$

Volume V generated by the curve

$= \pi\int_0^{\ln 4} x^2 e^{-2x} \, dx$

8 Let $u = x^2$ and $\frac{dv}{dx} = e^{-2x}$

So $\frac{du}{dx} = 2x$ and $v = -\frac{1}{2}e^{-2x}$

Integrating by parts

$$V = \pi\left(\left[-\frac{x^2e^{-2x}}{2}\right]_0^{\ln 4} + \frac{1}{2}\int_0^{\ln 4} 2xe^{-2x}\ dx\right)$$

$$= \pi\left(-\frac{(\ln 4)^2}{32} + \int_0^{\ln 4} xe^{-2x}\ dx\right)$$

Now let $u = x$ and $\frac{dv}{dx} = e^{-2x}$

So $\frac{du}{dx} = 1$ and $v = -\frac{1}{2}e^{-2x}$

Integrating by parts again

$$V = \pi\left(-\frac{(\ln 4)^2}{32} - \left[\frac{xe^{-2x}}{2}\right]_0^{\ln 4} + \frac{1}{2}\int_0^{\ln 4} e^{-2x}\ dx\right)$$

$$= \pi\left(-\frac{(\ln 4)^2}{32} - \frac{\ln 4}{32} - \frac{1}{4}\left[e^{-2x}\right]_0^{\ln 4}\right)$$

$$= \pi\left(-\frac{(\ln 4)^2}{32} - \frac{\ln 4}{32} - \frac{1}{4}\left(\frac{1}{16} - 1\right)\right)$$

$$= \pi\left(\frac{15}{64} - \frac{(\ln 4)^2}{32} - \frac{\ln 4}{32}\right)$$

$$= 0.4115$$

Volume generated by $y = \frac{1}{4}x$

between $x = 0$ and $x = \ln 4$ is the volume of a cone of height $\ln 4$ and radius $\frac{1}{4}\ln 4$

$$= \frac{1}{3}\pi\left(\frac{1}{4}\ln 4\right)^2 \times \ln 4$$

$$= \frac{\pi}{48}(\ln 4)^3$$

$$= 0.1744$$

Volume generated by

$R = 0.4115 - 0.1744 = 0.237$ (to 3 s.f.)

Challenge

The x coordinates of the points of intersection between the curve and the line are given by

$$\sin x = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4} \text{ and } \frac{3\pi}{4}$$

Volume generated when R is rotated around $y = \frac{1}{\sqrt{2}}$ will be

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\sin x - \frac{1}{\sqrt{2}}\right)^2 dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\sin^2 x - \frac{2}{\sqrt{2}}\sin x + \frac{1}{2}\right) dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{1}{2}(1 - \cos 2x) - \frac{2}{\sqrt{2}}\sin x + \frac{1}{2}\right) dx$$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(-\frac{1}{2}\cos 2x - \frac{2}{\sqrt{2}}\sin x + 1\right) dx$$

$$= \pi\left[-\frac{1}{4}\sin 2x + \sqrt{2}\cos x + x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \pi\left(\left(\frac{1}{4} + \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{3\pi}{4}\right) - \left(-\frac{1}{4} + \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}\right)\right)$$

$$= \pi\left(\frac{\pi}{2} - \frac{3}{2}\right)$$

$$= \frac{\pi(\pi - 3)}{2}$$