Methods in calculus 3D

1 Using $x = a\tan\theta, \mathrm{d}x = a\sec^2 \mathrm{d}\theta$

$$\text{so } \int \frac{1}{a^2 + x^2}\mathrm{d}x = \int \frac{1}{a^2 + a^2\tan^2\theta} a\sec^2\theta \;\mathrm{d}\theta$$
$$= \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\mathrm{d}\theta$$
$$= \frac{1}{a}\int \mathrm{d}\theta$$
$$= \frac{1}{a}\theta + C$$
$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$x = a\tan\theta \Rightarrow \arctan\left(\frac{x}{a}\right)$

2 Using $x = \cos\theta,\ \mathrm{d}x = -\sin\theta\,\mathrm{d}\theta$

$$\text{so } \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \int \frac{1}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\mathrm{d}\theta$$
$$= -\int \mathrm{d}\theta$$
$$= -\theta + C$$
$$= -\arccos x + C$$

3 a Let $x = 2\sin\theta$, so $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$

$$\int \frac{3}{\sqrt{4-x^2}}\mathrm{d}x = \int \frac{3}{\sqrt{4-4\sin^2\theta}} 2\cos\theta\,\mathrm{d}\theta$$
$$= \int \frac{6\cos\theta}{2\cos\theta}\mathrm{d}\theta$$
$$= 3\int \mathrm{d}\theta$$
$$= 3\theta + C$$
$$= 3\arcsin\left(\frac{x}{2}\right) + C$$

b Let $x = \sqrt{5}\tan\theta$, so $\mathrm{d}x = \sqrt{5}\sec^2\theta\,\mathrm{d}\theta$

$$\int \frac{4}{5+x^2}\mathrm{d}x = \int \frac{4}{5+5\tan^2\theta}\sqrt{5}\sec^2\theta\,\mathrm{d}\theta$$
$$= \int \frac{4\sqrt{5}\sec^2\theta}{5\sec^2\theta}\mathrm{d}\theta$$
$$= \frac{4\sqrt{5}}{5}\int \mathrm{d}\theta$$
$$= \frac{4\sqrt{5}}{5}\theta + C$$
$$= \frac{4\sqrt{5}}{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

$5 + 5\tan^2\theta = 5(1+\tan^2\theta) = 5\sec^2\theta$

3 c $\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right)+C$

Using $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C$

4 $\int \frac{1}{4+3x^2}\,dx$

$= \frac{1}{3}\int \frac{1}{\frac{4}{3}+x^2}\,dx$

$= \frac{1}{3}\left(\frac{1}{\sqrt{\frac{4}{3}}}\arctan\left(\frac{x}{\sqrt{\frac{4}{3}}}\right)\right)+c$

$= \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}x}{2}\right)+c$

Therefore, $A = \frac{\sqrt{3}}{6}$ and $B = \frac{\sqrt{3}}{2}$

5 $\int \frac{1}{\sqrt{3-4x^2}}\,dx$

$= \frac{1}{\sqrt{4}}\int \frac{1}{\sqrt{\frac{3}{4}-x^2}}\,dx$

$= \frac{1}{2}\arcsin\left(\frac{x}{\sqrt{\frac{3}{4}}}\right)$

$= \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{3}}\right)+c$

Therefore, $P = \frac{1}{2}$ and $Q = \frac{2}{\sqrt{3}}$

6 a $\int_1^3 \frac{2}{1+x^2}\,dx = 2[\arctan x]_1^3$

$= 2(\arctan 3 - \arctan 1)$

$= 2\left(\arctan 3 - \frac{\pi}{4}\right)$

Remember that you need to be in radian mode.

6 b $\int_{0.25}^{0.5} \frac{3}{\sqrt{1-4x^2}}\,dx = \frac{3}{\sqrt{4}}\int_{0.25}^{0.5} \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx$

$= \frac{3}{2}\left[\arcsin \frac{x}{\sqrt{\frac{1}{4}}}\right]_{0.25}^{0.5}$

$= \frac{3}{2}\left[\arcsin 2x\right]_{0.25}^{0.5}$

$= \frac{3}{2}(\arcsin 0.5 - \arcsin 0.25)$

$= \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$= \frac{\pi}{2}$

c $\int_{-1}^{2} \frac{1}{\sqrt{21-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int_{-1}^{2} \frac{1}{\sqrt{7-x^2}}\,dx$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{x}{\sqrt{7}}\right)\right]_{-1}^{2}$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{2}{\sqrt{7}}\right) - \arcsin\left(-\frac{1}{\sqrt{7}}\right)\right]$

$= \frac{1}{\sqrt{3}}[0.85707\ldots - (-0.38759\ldots)]$ ← You need to be in radian mode

$= 0.719 \quad (3\text{ s.f.})$

7 $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

$= \left[\arcsin\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

8 $f(x) = \frac{2+3x}{1+3x^2}$

$\int f(x)\,dx = \int \frac{2+3x}{1+3x^2}\,dx$

$= 2\int \frac{1}{1+3x^2}\,dx + 3\int \frac{x}{1+3x^2}\,dx$

Consider $\int \frac{1}{1+3x^2}\,dx$

$= \frac{1}{3}\int \frac{1}{\frac{1}{3} + x^2}\,dx$

8 $= \frac{1}{3}\left(\frac{1}{\sqrt{\frac{1}{3}}}\arctan\left(\frac{x}{\sqrt{\frac{1}{3}}}\right)\right) + c_1$

$= \frac{\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right) + c_1$

Similarly, consider $\int \frac{x}{1+3x^2}\,dx$

Let $u = 1+3x^2$ and $du = 6x\,dx$

$\int \frac{x}{1+3x^2}\,dx = \frac{1}{6}\int \frac{1}{u}\,du$

$= \frac{1}{6}\ln u + c_2$

$= \frac{1}{6}\ln\left(1+3x^2\right) + c_2$

Therefore,

$\int f\left(x\right)dx$

$= 2\int \frac{1}{1+3x^2}\,dx + 3\int \frac{x}{1+3x^2}\,dx$

$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right) + \frac{1}{2}\ln\left(1+3x^2\right) + c$

Therefore, $A = \frac{2\sqrt{3}}{3}$ and $B = \frac{1}{2}$

9 $f\left(x\right) = \frac{2x-1}{\sqrt{2-x^2}}$

$\int f\left(x\right)dx = \int \frac{2x-1}{\sqrt{2-x^2}}\,dx$

$= \int \frac{2x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{2-x^2}}\,dx$

Consider $\int \frac{2x}{\sqrt{2-x^2}}\,dx$

Let $u = 2-x^2$ and $du = -2x\,dx$

$\int \frac{2x}{\sqrt{2-x^2}}\,dx = -\int \frac{1}{\sqrt{u}}\,du$

$= -2\sqrt{u} + c_1$

$= -2\sqrt{2-x^2} + c_1$

Consider $\int \frac{1}{\sqrt{2-x^2}}\,dx$

$= \arcsin\left(\frac{x}{\sqrt{2}}\right) + c_2$

9 Therefore,

$\int f(x)\,dx$

$= \int \frac{2x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{2-x^2}}\,dx$

$= -2\sqrt{2-x^2} - \arcsin\left(\frac{x}{\sqrt{2}}\right) + c$

Therefore, $A = -1$ and $B = -2$

10 $f(x) = \frac{8x-3}{4+x^2}$

$\int f(x)\,dx = \int \frac{8x-3}{4+x^2}\,dx$

$= \int \frac{8x}{4+x^2}\,dx - 3\int \frac{1}{4+x^2}\,dx$

Consider $\int \frac{1}{4+x^2}\,dx$

$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + c_1$

Similarly, consider $\int \frac{8x}{4+x^2}\,dx$

Let $u = 4 + x^2$ and $du = 2x dx$

$\int \frac{8x}{4+x^2}\,dx = 4\int \frac{1}{u}\,du$

$= 4\ln u + c_2$

$= 4\ln(4+x^2) + c_2$

Therefore,

$\int f(x)\,dx$

$= \int \frac{8x}{4+x^2}\,dx - 3\int \frac{1}{4+x^2}\,dx$

$= 4\ln(4+x^2) - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c$

Therefore, $A = 4$ and $B = -\frac{3}{2}$

11 $f(x) = \frac{4x-1}{\sqrt{6-5x^2}}$

$\int f(x)\,dx = \int \frac{4x-1}{\sqrt{6-5x^2}}\,dx$

$= \int \frac{4x}{\sqrt{6-5x^2}}\,dx - \int \frac{1}{\sqrt{6-5x^2}}\,dx$

11 Consider $\int \frac{1}{\sqrt{6-5x^2}}\,dx$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\frac{6}{5}-x^2}}\,dx$$

$$= \frac{1}{\sqrt{5}}\arcsin\left(\frac{x}{\sqrt{\frac{6}{5}}}\right)+c_1$$

$$= \frac{1}{\sqrt{5}}\arcsin\left(\frac{\sqrt{5}x}{\sqrt{6}}\right)+c_1$$

Similarly, consider $\int \frac{4x}{\sqrt{6-5x^2}}\,dx$

Let $u = 6-5x^2$ and $du = -10x\,dx$

$$\int \frac{4x}{\sqrt{6-5x^2}}\,dx = -\frac{2}{5}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{4}{5}\sqrt{u}+c_2$$

$$= -\frac{4}{5}\sqrt{6-5x^2}+c_2$$

Therefore,

$\int f(x)\,dx$

$$= \int \frac{4x}{\sqrt{6-5x^2}}\,dx - \int \frac{1}{\sqrt{6-5x^2}}\,dx$$

$$= -\frac{4}{5}\sqrt{6-5x^2} - \frac{1}{\sqrt{5}}\arcsin\left(\frac{\sqrt{5}x}{\sqrt{6}}\right)+c$$

Therefore, $P = -\frac{4}{5}$ and $Q = -\frac{1}{\sqrt{5}}$

12 a $f(x) = \frac{x+5}{x^2+16}$

$$\int f(x)\,dx = \int \frac{x+5}{x^2+16}\,dx$$

$$= \int \frac{x}{x^2+16}\,dx + 5\int \frac{1}{x^2+16}\,dx$$

Consider $\int \frac{x}{x^2+16}\,dx$

Let $u = x^2+16$ and $du = 2x\,dx$

$$\int \frac{x}{x^2+16}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$

$= \frac{1}{2}\ln u + c_1$

$= \frac{1}{2}\ln\left(x^2 + 16\right) + c_1$

12 a Similarly, consider $\int \frac{1}{x^2+16}\,dx$

$= \frac{1}{4}\arctan\left(\frac{x}{4}\right) + c_1$

Therefore,

$\int f\left(x\right)dx$

$= \int \frac{x}{x^2+16}\,dx + 5\int \frac{1}{x^2+16}\,dx$

$= \frac{1}{2}\ln\left(x^2+16\right) + \frac{5}{4}\arctan\left(\frac{x}{4}\right) + c$

Therefore, $A = \frac{1}{2}$ and $B = \frac{5}{4}$

b The mean value of $f\left(x\right)$ over the interval $\left[0,4\right]$ is:

$\frac{1}{4}\int_0^4 f\left(x\right)dx$

$= \frac{1}{4}\left[\frac{1}{2}\ln\left(x^2+16\right) + \frac{5}{4}\arctan\left(\frac{x}{4}\right)\right]_0^4$

$= \frac{1}{4}\left(\frac{1}{2}\ln 2 + \frac{5\pi}{16}\right)$

c The mean value of $-4f\left(x\right)$ over the interval $\left[0,4\right]$ is -4 times the mean value of $f\left(x\right)$ over the interval $\left[0,4\right]$

$= -\frac{1}{2}\ln 2 - \frac{5\pi}{16}$

13 With $x = \frac{2}{3}\tan\theta$ and $dx = \frac{2}{3}\sec^2\theta\,d\theta$,

$9x^2 + 4 = 9\left(\frac{4}{9}\tan^2\theta\right) + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$

and $\frac{x^2}{9x^2+4} = \frac{\frac{4}{9}\tan^2\theta}{4\sec^2\theta} = \frac{\tan^2\theta}{9\sec^2\theta}$

13 so $\int \frac{x^2}{9x^2+4}\,dx = \int \frac{\tan^2\theta}{9\sec^2\theta} \times \frac{2}{3}\sec^2\theta\, d\theta$

$$= \frac{2}{27}\int \tan^2\theta\, d\theta$$

$$= \frac{2}{27}\int (\sec^2\theta - 1)\, d\theta$$

$$= \frac{2}{27}(\tan\theta - \theta) + C$$

$$= \frac{2}{27}\left(\frac{3x}{2} - \arctan\frac{3x}{2}\right) + C$$

$$= \frac{x}{9} - \frac{2}{27}\arctan\frac{3x}{2} + C$$

14 With $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\, d\theta$

$1-4x^2 = 1-\sin^2\theta = \cos^2\theta$ and so $\frac{x^2}{\sqrt{1-4x^2}} = \frac{\sin^2\theta}{4\cos\theta}$

So $\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{4\cos\theta} \times \frac{1}{2}\cos\theta\, d\theta$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$$

$$= \frac{1}{16}\int_0^{\frac{\pi}{6}} (1 - 2\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{16}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{192}(2\pi - 3\sqrt{3})$$

Challenge

a $\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x$

$x = \sec\theta$, so $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$\int \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta$

$= \int d\theta = \theta + c = \text{arcsec}x + c$

b $\int \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$

$x = \sec\theta$, so $\mathrm{d}x = \sec\theta\tan\theta\,d\theta$

Substituting gives

$x = \sec\theta$ so $\cos\theta = \frac{1}{x}$ and $\tan\theta = \sqrt{x^2-1}$

Therefore

$\tan\theta - \theta + c = \sqrt{x^2-1} - \text{arcsec}\,x + c$