Methods in calculus 3C

1 a Let $y = \arctan x$

then $\tan y = x$

$\sec^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

But $\tan y = x$, so $\cos y = \frac{1}{\sqrt{1+x^2}}$

Therefore,

$\frac{dy}{dx} = \frac{1}{1+x^2}$

b Let $y = \arccos x$

then $\cos y = x$

$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = -\frac{1}{\sin y}$

But $\cos y = x$, so $\sin y = \sqrt{1-x^2}$

Therefore,

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

c Let $y = \arccos x^2$

then $\cos y = x^2$

$-\sin y \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = -\frac{2x}{\sin y}$

But $\cos y = x^2$, so $\sin y = \sqrt{1-x^4}$

Therefore,

$\frac{dy}{dx} = -\frac{2x}{\sqrt{1-x^4}}$

d Let $y = \arctan\left(x^3 + 3x\right)$

then $\tan y = x^3 + 3x$

$\sec^2 y \frac{dy}{dx} = 3x^2 + 3$

$\frac{dy}{dx} = \frac{3x^2+3}{\sec^2 y}$

$= \left(3x^2+3\right)\cos^2 y$

But $\tan y = x^3 + 3x$

1 d So, $\cos y = \dfrac{1}{\sqrt{1+\left(x^3+3x\right)^2}}$

Therefore,

$$\frac{dy}{dx} = \frac{3x^2+3}{1+\left(x^3+3x\right)^2}$$

e Let $y = \arcsin\left(\dfrac{1}{x}\right)$

then $\sin y = \dfrac{1}{x}$

$$\cos y \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2 \cos y}$$

But $\sin y = \dfrac{1}{x}$

So, $\cos y = \sqrt{1-\left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{x^2-1}}{x}$

Therefore,

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

2 $y = (\arccos x)(\arcsin x)$

Using the product rule for differentiating, and the standard results for $\dfrac{d}{dx}(\arccos x)$ and $\dfrac{d}{dx}(\arcsin x)$

,

$$\frac{dy}{dx} = \left((\arccos x)\frac{d}{dx}(\arcsin x) + (\arcsin x)\frac{d}{dx}(\arccos x) \right)$$

$$\frac{dy}{dx} = \arccos x . \frac{1}{\sqrt{1-x^2}} - \arcsin x . \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{\arccos x - \arcsin x}{\sqrt{1-x^2}}$$

3 $y = \dfrac{1+\arctan x}{1-\arctan x}$

$y(1-\arctan x) = 1+\arctan x$

$(1+y)\arctan x = y-1$

$\arctan x = \dfrac{y-1}{y+1}$

$\tan\left(\dfrac{y-1}{y+1}\right) = x$

$\sec^2\left(\dfrac{y-1}{y+1}\right)\left(\dfrac{(y+1)-(y-1)}{(y+1)^2}\right)\dfrac{dy}{dx} = 1$

$\left(1+\tan^2\left(\dfrac{y-1}{y+1}\right)\right)\left(\dfrac{2}{(y+1)^2}\right)\dfrac{dy}{dx} = 1$

$(1+x^2)\dfrac{(1-\arctan x)^2}{2}\dfrac{dy}{dx} = 1$

$\dfrac{dy}{dx} = \dfrac{2}{(1+x^2)(1-\arctan x)^2}$

4 $f(x) = \arccos x + \arcsin x$

$f'(x) = \dfrac{d}{dx}(\arccos x) + \dfrac{d}{dx}(\arcsin x)$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0$

$f(x) = \int f'(x)dx = \int 0 dx = C$

Consider

$f(0) = \arccos 0 + \arcsin 0$

$= \dfrac{\pi}{2} + 0 = \dfrac{\pi}{2} = C$

Therefore, $f(x) = \dfrac{\pi}{2}$ for all values of x.

5 **a** Let $y = \text{arc}\cos 2x$

Let $t = 2x \quad y = \text{arcos}\, t$

then $\dfrac{dt}{dx} = 2 \quad \dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}}$

$\dfrac{dy}{dt} = \dfrac{-1}{\sqrt{1-t^2}} \times 2$

$= \dfrac{-2}{\sqrt{1-4x^2}}$

5 b Let $y = \arctan\frac{x}{2}$

Let $t = \frac{x}{2}$ $\quad y = \arctan t$

$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{1+t^2}$

$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{1}{2} = \frac{1}{2\left(1+\frac{x^2}{4}\right)} = \frac{2}{4+x^2}$ or $\frac{2}{x^2+4}$

c Let $y = \arcsin 3x$

$\sin y = 3x$

$\cos y \frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{\cos y} = \frac{3}{\sqrt{1-\sin^2 y}}$

$= \frac{3}{\sqrt{1-9x^2}}$

$= \frac{3}{\sqrt{1-9x^2}}$

d Let $y = \text{arccot}(x+1)$

Let $t = x+1$ and $\frac{dt}{dx} = 1$

$y = \text{arccot}\, t$

$\cot y = t$

$-\text{cosec}^2 y \frac{dy}{dt} = 1$

$\frac{dy}{dt} = -\sin^2 y$

$= -\frac{1}{1+(x+1)^2}$

Using chain rule,

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -\frac{1}{1+(x+1)^2}$

5 e Let $y = \arcsin\left(1-x^2\right)$

Let $t = 1-x^2$ and $\frac{dt}{dx} = -2x$

$y = \arcsin t$

$\sin y = t$

$\cos y \frac{dy}{dt} = 1$

$$\frac{dy}{dt} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-\left(1-x^2\right)^2}}$$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= -\frac{2x}{\sqrt{1-\left(1-x^2\right)^2}}$$

$$= -\frac{2x}{\sqrt{x^2\left(2-x^2\right)}}$$

f Let $y = \arccos x^2$

Let

$t = x^2 \qquad y = \arccos t$

$\frac{dt}{dx} = 2x \qquad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-t^2}} \times 2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

g Let $y = e^x \arccos x$

$$\frac{dy}{dx} = e^x \arccos x - e^x \frac{1}{\sqrt{1-x^2}}$$

$$= e^x \left(\arccos x - \frac{1}{\sqrt{1-x^2}}\right)$$

5 h Let $y = \arcsin x \cos x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\cos x + \arcsin x \times -\sin x$$

$$= \frac{\cos x}{\sqrt{1-x^2}} - \sin x \arcsin x$$

i Let $y = x^2 \arccos x$

$$\frac{dy}{dx} = 2x\arccos x - x^2 \times \frac{1}{\sqrt{1-x^2}}$$

$$= 2x\arccos x - \frac{x^2}{\sqrt{1-x^2}}$$

$$= x\left(2\arccos x - \frac{x}{\sqrt{1-x^2}}\right)$$

j Let $y = e^{\arctan x}$

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

6

$$\tan y = x \arctan x$$

$$\sec^2 y \frac{dy}{dx} = \arctan x + \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(\arctan x + \frac{x}{1+x^2}\right)$$

$$= \frac{1}{1+x^2(\arctan x)^2}\left(\arctan x + \frac{x}{1+x^2}\right)$$

7 $y = \arcsin x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{0 - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{x(1-x^2)^{-\frac{1}{2}}}{(1-x^2)}$$

$$= \frac{x}{\sqrt{1-x^2}\,(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

8 $y = \arcsin 2x \quad x = \frac{1}{4} \quad y = \arcsin x\left(\frac{2}{4}\right) = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{4}{\sqrt{3}}$$

Tangent is

$$\left(y - \frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$$

$$\sqrt{3}y - \frac{\pi\sqrt{3}}{6} = 4x - 1$$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{6} - \frac{1}{\sqrt{3}}$$

9 **a** $y = (\arctan x)^2$

Let $t = \arctan x$

So, $\frac{dt}{dx} = \frac{1}{1+x^2}$

$y = t^2$

$$\frac{dy}{dt} = 2t = 2\arctan x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times, \frac{dt}{dx}$$

$$= \frac{2\arctan x}{1+x^2}$$

9 b $y = \dfrac{1}{\arcsin x}$

Let $t = \arcsin x$

So, $\dfrac{dt}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

$y = \dfrac{1}{t}$

$\dfrac{dy}{dt} = -\dfrac{1}{t^2} = -\dfrac{1}{(\arcsin x)^2}$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times, \dfrac{dt}{dx}$

$= -\dfrac{1}{\sqrt{1-x^2}(\arcsin x)^2}$.

c $y = \arctan(\arctan x)$

Let $t = \arctan x$

So, $\dfrac{dt}{dx} = \dfrac{1}{1+x^2}$

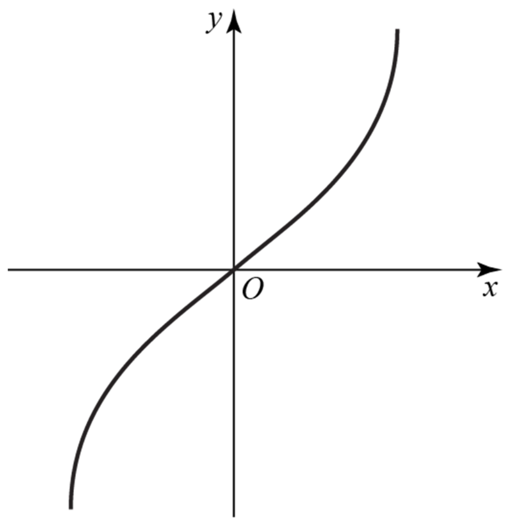
$y = \arctan t$

$\dfrac{dy}{dt} = \dfrac{1}{1+t^2} = \dfrac{1}{1+(\arctan x)^2}$

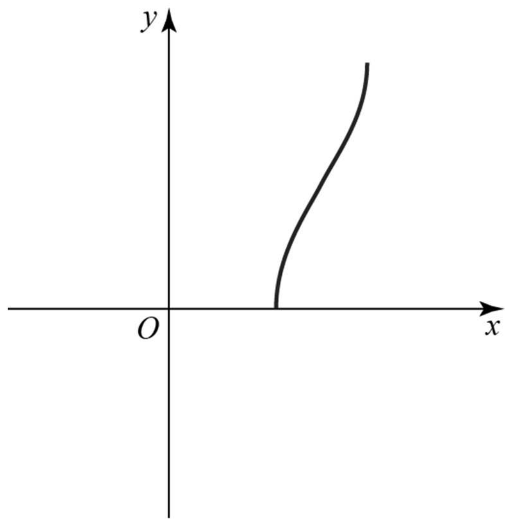
$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$= \dfrac{1}{(1+x^2)(1+(\arctan x)^2)}$

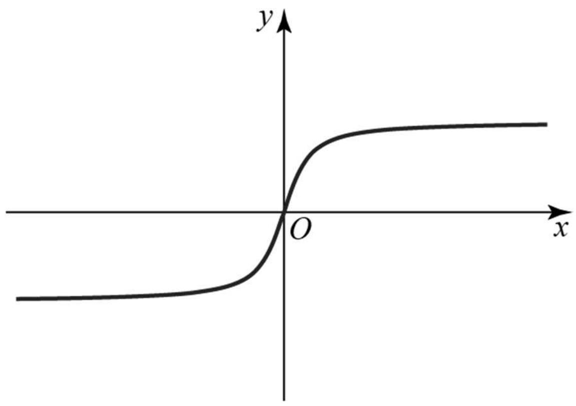
10 a The graph of $y = \arcsin(\arcsin x)$ is:



10 b The graph of $y = \arccos(\arccos x)$ is:



c The graph of $y = \arctan(\arctan x)$ is:



11 a Let $t = \arccos x$, so $\cos t = x$

$$\sin(\arccos x) = \sin t$$
$$= \pm\sqrt{1-\cos^2 t}$$

Since $\arccos x$ has the range $[0, \pi]$, and $\sin t$ is positive on this domain, we have

$$\sin(\arccos x) = \sqrt{1-x^2}$$

b Let $t = \arctan x$, so $\tan t = x$

Therefore,

$$\cos(\arctan x) = \cos t$$
$$= \frac{1}{\sec t}$$
$$= \pm\frac{1}{\sqrt{1+x^2}}$$

Since $\arctan x$ has the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\cos t$ is positive on this domain, we have

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

11 c Let $t = \arccos x$, so $\cos t = x$

Therefore,

$$\sec(\arccos x) = \sec t$$
$$= \frac{1}{\cos t}$$
$$= \frac{1}{x}$$

d Let $t = \text{arcsec}x$, so $\sec t = x$

Therefore,

$$\sin(\text{arcsec}x) = \sin t$$
$$= \pm\sqrt{1-\cos^2 t}\,.$$
$$= \pm\sqrt{1-\left(\frac{1}{x}\right)^2}$$
$$= \pm\frac{\sqrt{x^2-1}}{x}$$

Since $\text{arcsec}x$ has the range $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$, and $\sin t$ is positive on this domain, we have

$$\sin(\text{arcsec}x) = \sqrt{1-\frac{1}{x^2}}$$