Methods in calculus 3B

1 a $f(x)=1$

The mean value of $f(x)$ on $[0,1]$ is:

$$\frac{1}{1-0}\int_0^1 1\,dx=[x]_0^1=1$$

b $f(x)=\frac{1}{x+1}$

The mean value of $f(x)$ on $[0,1]$ is:

$$\frac{1}{1-0}\int_0^1 \frac{1}{x+1}dx=\left[\ln(x+1)\right]_0^1$$
$$=\ln 2-\ln 1=\ln 2$$

c $f(x)=e^x+1$

The mean value of $f(x)$ on $[0,1]$ is:

$$\frac{1}{1-0}\int_0^1 e^x+1\,dx=\left[e^x+x\right]_0^1$$
$$=e+1-1=e$$

2 a $f(x)=\frac{e^{3x}}{e^{3x}+1}$

Consider $\int\frac{e^{3x}}{e^{3x}+1}dx$

Let $u=e^{3x}$ and $du=3e^{3x}dx$

$$\int\frac{e^{3x}}{e^{3x}+1}dx=\frac{1}{3}\int\frac{1}{u+1}du.$$
$$=\frac{1}{3}\ln(u+1)+C$$
$$=\frac{1}{3}\ln\left(e^{3x}+1\right)+C$$

The mean value of $f(x)$ on $[0,2]$ is:

$$\frac{1}{2}\int_0^2\frac{e^{3x}}{e^{3x}+1}dx$$
$$=\frac{1}{6}\left[\ln\left(e^{3x}+1\right)\right]_0^2$$
$$=\frac{1}{6}\left(\ln\left(e^6+1\right)-\ln 2\right)$$
$$=\frac{1}{6}\ln\left(\frac{e^6+1}{2}\right)$$

2 b $f(x)=\cos^3 x\sin^2 x$

Consider $\cos^3 x\sin^2 x$

$$=\cos x(\cos x\sin x)^2$$
$$=\cos x\left(\frac{1}{2}\sin 2x\right)^2$$
$$=\frac{1}{4}\cos x(\sin 2x\sin 2x)$$
$$=\frac{1}{4}\cos x\left(\frac{1-\cos 4x}{2}\right)$$
$$=\frac{1}{8}(\cos x-\cos x\cos 4x)$$
$$=\frac{1}{8}\left(\cos x-\frac{1}{2}\cos 5x-\frac{1}{2}\cos 3x\right)$$
$$=\frac{1}{16}(2\cos x-\cos 5x-\cos 3x)$$

Therefore,

$$\int_0^{\frac{\pi}{2}}\cos^3 x\sin^2 x\,dx$$
$$=\frac{1}{16}\int_0^{\frac{\pi}{2}}(2\cos x-\cos 5x-\cos 3x)dx$$
$$=\frac{1}{16}\left[2\sin x-\frac{1}{5}\sin 5x-\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{2}}$$
$$=\frac{1}{16}\left(2-\frac{1}{5}+\frac{1}{3}\right)=\frac{2}{15}$$

So the mean value of $f(x)$ on $\left[0,\frac{\pi}{2}\right]$ is:

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos^3 x\sin^2 x\,dx=\frac{4}{15\pi}$$

2 c $f(x) = xe^{-x}$

Consider $\int xe^{-x}$

Integrating by parts,

$$\int xe^{-x}\mathrm{d}x = -xe^{-x} + \int e^{-x}\mathrm{d}x$$
$$= -xe^{-x} - e^{-x}$$

So the mean value of $\mathrm{f}(x)$ on $[1,3]$ is:

$$\frac{1}{3-1}\int_1^3 xe^{-x}\mathrm{d}x$$
$$= \frac{1}{2}\left[-xe^{-x} - e^{-x}\right]_1^3$$
$$= \frac{1}{2}\left(-3e^{-3} - e^{-3} + e^{-1} + e^{-1}\right)$$
$$= \frac{1}{2}\left(-\frac{4}{e^3} + \frac{2}{e^1}\right) = \frac{e^2-2}{e^3}$$

d $\mathrm{f}(x) = \dfrac{5}{(x+2)(2x+1)}$

Consider $\displaystyle\int_0^3 \frac{5}{(x+2)(2x+1)}\mathrm{d}x$

$$= \int_0^3 \left(-\frac{5}{3(x+2)} + \frac{10}{3(2x+1)}\right)\mathrm{d}x$$
$$= \frac{5}{3}\left[-\ln(|x+2|) + \ln(|2x+1|)\right]_0^3$$
$$= \frac{5}{3}\left[\ln\left(\left|\frac{2x+1}{x+2}\right|\right)\right]_0^3$$
$$= \frac{5}{3}\left(\ln\frac{7}{5} - \ln\frac{1}{2}\right) = \frac{5}{3}\ln\frac{14}{5}$$

So the mean value of $\mathrm{f}(x)$ on $[0,3]$ is:

$$\frac{1}{3-0}\int_0^3 \frac{5}{(x+2)(2x+1)}\mathrm{d}x$$
$$= \frac{5}{9}\ln\frac{14}{5}$$

2 e $\mathrm{f}(x) = (\sec x - \cos x)^2$

Consider $\int (\sec x - \cos x)^2\,\mathrm{d}x$

$$= \int \left(\sec^2 x - 2\sec x\cos x + \cos^2 x\right)\mathrm{d}x$$
$$= \int \sec^2 x\mathrm{d}x - 2\int \mathrm{d}x + \int\left(\frac{\cos 2x + 1}{2}\right)\mathrm{d}x$$
$$= \tan x - 2x + \frac{1}{4}\sin 2x + \frac{1}{2}x + C$$
$$= \tan x + \frac{1}{4}\sin 2x - \frac{3}{2}x + C$$

$$\int_0^{\frac{\pi}{4}} (\sec x - \cos x)^2\,\mathrm{d}x$$
$$= \left[\tan x + \frac{1}{4}\sin 2x - \frac{3}{2}x\right]_0^{\frac{\pi}{4}}$$
$$= 1 + \frac{1}{4} - \frac{3\pi}{8} = \frac{5}{4} - \frac{3\pi}{8}$$

So the mean value of $\mathrm{f}(x)$ on $\left[0, \frac{\pi}{4}\right]$ is:

$$\frac{4}{\pi}\int_0^{\frac{\pi}{4}} (\sec x - \cos x)^2\,\mathrm{d}x = \frac{5}{\pi} - \frac{3}{2}$$

3 a For turning points, $\mathrm{f}'(x) = 0$

$$\mathrm{f}'(x) = 3x^2 - 6x - 24 = 0$$
$$3(x^2 - 2x - 8) = 0$$
$$3(x-4)(x+2) = 0$$
$$x = 4, -2$$

When $x = 4$,

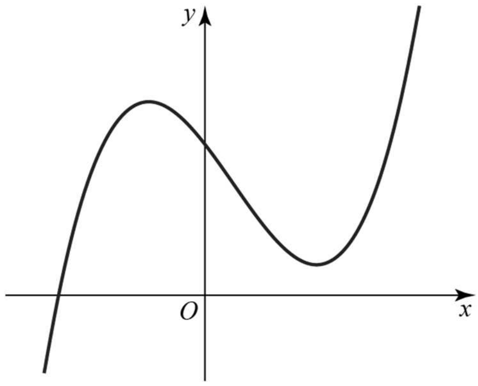
$$\mathrm{f}(x) = 4^3 - 3\times 4^2 - 24\times 4 + 100$$
$$\mathrm{f}(x) = 20$$

When $x = -2$,

$$\mathrm{f}(x) = (-2)^3 - 3\times(-2)^2 - 24\times(-2) + 100$$
$$\mathrm{f}(x) = 128$$

Therefore, the turning points are $(4, 20)$ and $(-2, 128)$.

3 b The graph of
$y = \mathrm{f}(x) = x^3 - 3x^2 - 24x + 100$ is:



c The lower bound on the mean value is 20 and the upper bound is 128 since these are the minimum and maximum values of the function on the interval $[-2, 4]$.

d The mean value of $\mathrm{f}(x)$ over the interval $[-2, 4]$ is:

$$\frac{1}{4-(-2)}\int_{-2}^{4}\mathrm{f}(x)\,\mathrm{d}x$$
$$= \frac{1}{6}\int_{-2}^{4}\left(x^3 - 3x^2 - 24x + 100\right)\mathrm{d}x$$
$$= \frac{1}{6}\left[\frac{x^4}{4} - \frac{3x^3}{3} - \frac{24x^2}{2} + 100x\right]_{-2}^{4}$$
$$= \frac{1}{6}\begin{pmatrix}64 - 64 - 192 + 400 - \\ 4 - 8 + 48 + 200\end{pmatrix}$$
$$= 74$$

4 $\mathrm{f}(x) = \dfrac{\sin x \cos x}{\cos 2x + 2}$

Consider $\int \dfrac{\sin x \cos x}{\cos 2x + 2}\mathrm{d}x$

$$= \frac{1}{2}\int \frac{\sin 2x}{\cos 2x + 2}\mathrm{d}x$$

Let $u = \cos 2x + 2$ and $du = -2\sin 2x\mathrm{d}x$. So,

$$\frac{1}{2}\int \frac{\sin 2x}{\cos 2x + 2}\mathrm{d}x = -\frac{1}{4}\int \frac{1}{u}du$$
$$= -\frac{1}{4}\ln u + C$$
$$= -\frac{1}{4}\ln\left|\cos 2x + 2\right| + C$$

So the mean value of $\mathrm{f}(x)$ on $\left[0, \dfrac{\pi}{2}\right]$ is:

$$\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{\sin x \cos x}{\cos 2x + 2}\mathrm{d}x$$
$$= \frac{2}{\pi}\left[-\frac{1}{4}\ln\left|\cos 2x + 2\right|\right]_{0}^{\frac{\pi}{2}}.$$
$$= \frac{\ln 3}{2\pi}$$

5 $\mathrm{f}(x) = x\sqrt{x+4}$

Consider $\int x\sqrt{x+4}\mathrm{d}x$

$$= \int (x + 4 - 4)\sqrt{x+4}\mathrm{d}x$$
$$= \int (x+4)^{\frac{3}{2}}\mathrm{d}x - 4\int\sqrt{x+4}\mathrm{d}x$$
$$= \frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}} + C$$

So the mean value of on $[0, 5]$ is:

$$\frac{1}{5}\int_{0}^{5}x\sqrt{x+4}\mathrm{d}x$$
$$= \frac{1}{5}\left[\frac{2}{5}(x+4)^{\frac{5}{2}} - \frac{8}{3}(x+4)^{\frac{3}{2}}\right]_{0}^{5}$$
$$= \frac{1}{5}\left(\frac{486}{5} - 72 - \frac{64}{5} + \frac{64}{3}\right)$$
$$= \frac{506}{75}$$

6 $f(x) = x\sin 2x$

Consider $\int x\sin 2x\,dx$

Integrating by parts,

$$= -x\frac{\cos 2x}{2} + \int \frac{\cos 2x}{2}\,dx$$

$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

So the mean value of $f(x)$ on $\left[0, \frac{\pi}{3}\right]$ is:

$$\frac{3}{\pi}\int_0^{\frac{\pi}{3}} x\sin 2x\,dx$$

$$= \frac{3}{\pi}\left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{3}{\pi}\left(\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)$$

$$= \frac{1}{4} + \frac{3\sqrt{3}}{8\pi}$$

7 a $f(x) = \frac{5x}{(2x-1)(x+2)}$

Consider $\int \frac{5x}{(2x-1)(x+2)}\,dx$

$$= \int\left(\frac{1}{2x-1} + \frac{2}{x+2}\right)dx$$

$$= \int \frac{1}{2x-1}\,dx + \int \frac{2}{x+2}\,dx$$

$$= \frac{1}{2}\ln|2x-1| + 2\ln|x+2| + C$$

$$= \ln\left|\sqrt{2x+1}\,(x+2)^2\right| + C$$

So the mean value of $f(x)$ on $[1, 5]$ is:

$$\frac{1}{4}\int_1^5 \frac{5x}{(2x-1)(x+2)}\,dx$$

$$= \frac{1}{4}\left[\ln\left|\sqrt{2x+1}\,(x+2)^2\right|\right]_1^5$$

$$= \frac{1}{4}\left(\ln(3\times 49) - \ln 9\right)$$

$$= \frac{1}{4}\ln\frac{49}{3}$$

7 b The mean value of $f(x) + \ln k$ is equal to the mean value of $f(x)$ plus the mean value of $\ln k$.

The mean value of $\ln k$ over the interval $[1, 5]$ is:

$$\frac{1}{4}\int_1^5 \ln k\,dx = \frac{1}{4}\ln k\int_1^5 dx$$

$$= \frac{1}{4}\ln k\,[x]_1^5 = \frac{1}{4}\ln k^4$$

Therefore, the mean value of $f(x) + \ln k$ is $\frac{1}{4}\ln\frac{49}{3} + \frac{1}{4}\ln k^4 = \frac{1}{4}\ln\frac{49k^4}{3}$

8 a $f(x) = x(x^2 - 4)^4$

Consider $\int x(x^2-4)^4\,dx$

Let $u = x^2 - 4$ and $du = 2x\,dx$

$$= \frac{1}{2}\int u^4\,dx$$

$$= \frac{u^5}{10} + C$$

$$= \frac{(x^2-4)^5}{10} + C$$

So the mean value of $f(x)$ on $[0, 2]$ is:

$$\frac{1}{2}\int_0^2 x(x^2-4)^4\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2-4)^5}{10}\right]_0^2$$

$$= \frac{256}{5}$$

8 b The mean value of $-2f(x)$ is equal to the mean value of $f(x)$ over the interval $[0, 2]$ multiplied by -2.

Therefore, the mean value of $-2f(x)$ on $[0, 2]$ is $-\frac{512}{5}$.

9 $\mathrm{f}(x)=\ln(kx)$

Consider $\int \ln(kx)\,\mathrm{d}x$

Integrating by parts,

$= x\ln(kx) - \int x\frac{1}{x}\,\mathrm{d}x$

$= x\ln(kx) - x + C$

So the mean value of $\mathrm{f}(x)$ on $[0,2]$ is:

$\frac{1}{2}\int_0^2 \ln(kx)\,\mathrm{d}x = -2$

$\frac{1}{2}\left[x\ln(kx)-x\right]_0^2 = -2$

$\frac{1}{2}(2\ln 2k - 2) = -2$

$\ln 2k = -1$

$2k = \frac{1}{e}$

$k = \frac{1}{2e}$

10 Given that the mean value of $\mathrm{f}(x)$ on the interval $[a,b]$ is m:

$\frac{1}{b-a}\int_a^b \mathrm{f}(x)\,\mathrm{d}x = m$

The mean value of $\mathrm{f}(x)+c$ on the interval $[a,b]$ is:

$\frac{1}{b-a}\int_a^b (\mathrm{f}(x)+c)\,\mathrm{d}x$

$= \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x + \frac{1}{b-a}\int_a^b c\,\mathrm{d}x$

$= \frac{1}{b-a}\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \frac{c}{b-a}\int_a^b \mathrm{d}x$

$= m + \frac{c}{b-a}(b-a) = m + c$

11 $\mathrm{f}(x) = \frac{1}{\sqrt{2-x}}$

Consider $\int \frac{1}{\sqrt{2-x}}\,\mathrm{d}x$

$= -2\sqrt{2-x} + C$

So the mean value of $\mathrm{f}(x)$ on $[0,2]$ is:

$\frac{1}{2}\int_0^2 \frac{1}{\sqrt{2-x}}\,\mathrm{d}x$

$= \frac{1}{2}\left[-2\sqrt{2-x}\right]_0^2$

$= \sqrt{2}$

12 The graph of $y=\mathrm{f}(x)=\sin^5 x$ on the interval $[0,\pi]$ is the negative of the graph graph of $y=\sin^5 x$ on the interval $[\pi,2\pi]$. Therefore, on evaluating the integral $\int_0^{2\pi}\sin^5 x\,\mathrm{d}x = \int_0^{\pi}\sin^5 x\,\mathrm{d}x + \int_{\pi}^{2\pi}\sin^5 x\,\mathrm{d}x$, the two integrals on the right hand side cancel each other out and the mean value of $\mathrm{f}(x)$ on the interval $[0,2\pi]$ is zero.

13 a $\int \mathrm{f}(x)\,\mathrm{d}x = \int \frac{\cos x}{(2+\sin x)^2}\,\mathrm{d}x$

Let $u = 2+\sin x$ and $du = \cos x\,\mathrm{d}x$. So,

$\int \frac{\cos x}{(2+\sin x)^2}\,\mathrm{d}x = \int \frac{1}{u^2}\,du$

$= -\frac{1}{u} + C = -\frac{1}{2+\sin x} + C$

b The mean value of $\mathrm{f}(x)$ over the interval $\left[0,\frac{5\pi}{3}\right]$ is:

$\frac{3}{5\pi}\int_0^{\frac{5\pi}{3}} \mathrm{f}(x)\,\mathrm{d}x$

$= \frac{3}{5\pi}\left[-\frac{1}{2+\sin x}\right]_0^{\frac{5\pi}{3}}$

$= \frac{3}{5\pi}\left(-\frac{8+2\sqrt{3}}{13}+\frac{1}{2}\right)$

$= -\frac{3}{130\pi}\left(3+4\sqrt{3}\right)$

13 c The mean value of $f(x)+3x$ is equal to the mean value of $f(x)$ plus the mean value of $3x$ on the given interval.
The mean value of $3x$ over the interval $\left[0,\frac{5\pi}{3}\right]$ is:

$$\frac{3}{5\pi}\int_0^{\frac{5\pi}{3}} 3x\,dx$$

$$=\frac{3}{5\pi}\left[\frac{3x^2}{2}\right]_0^{\frac{5\pi}{3}}$$

$$=\frac{5\pi}{2}$$

Therefore, the mean value of $f(x)+3x$ is

$$-\frac{3}{130\pi}\left(3+4\sqrt{3}\right)+\frac{5\pi}{2}$$

14 a For turning points, $f'(x)=0$

$$f'(x)=-3-4x=0$$

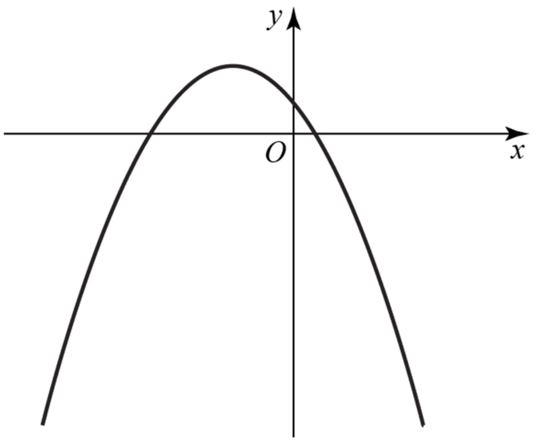
$$x=-\frac{3}{4}$$

At $x=-\frac{3}{4}$,

$$f(x)=1-3\left(-\frac{3}{4}\right)-2\left(-\frac{3}{4}\right)^2=\frac{17}{8}$$

So the turning point is $\left(-\frac{3}{4},\frac{17}{8}\right)$

The graph of $y=f(x)=1-3x-2x^2$ is:



14 b $\int_a^{a+1} f(x)\,dx=\int_a^{a+1}\left(1-3x-2x^2\right)dx$

$$=\left[x-\frac{3x^2}{2}-\frac{2x^3}{3}\right]_a^{a+1}$$

$$=\left(a+1-\frac{3(a+1)^2}{2}-\frac{2(a+1)^3}{3} - a+\frac{3a^2}{2}+\frac{2a^3}{3}\right)$$

$$=1-3a-\frac{3}{2}-2a^2-2a-\frac{2}{3}$$

$$=-\frac{7}{6}-5a-2a^2$$

b The mean value over an interval of length 1, say $[a,a+1]$, is given by

$$\int_a^{a+1} f(x)\,dx=-\frac{7}{6}-5a-2a^2$$

c To find the maximum value,

$$\frac{d}{da}\left(-\frac{7}{6}-5a-2a^2\right)=0$$

$$-5-4a=0$$

$$a=-\frac{5}{4}$$

The maximum value occurs when $a=-\frac{5}{4}$ and the maximum mean value is

$$=-\frac{7}{6}-5\left(-\frac{5}{4}\right)-2\left(-\frac{5}{4}\right)^2=\frac{47}{24}$$