Methods in calculus 3A

1 a $\int_1^\infty \frac{1}{x^3}\,dx$

$= \lim_{t\to\infty}\int_1^t \frac{1}{x^3}\,dx$

$= \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_1^t$

$= \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right)=\frac{1}{2}$

So $\int_1^\infty \frac{1}{x^3}\,dx$ converges and $\int_1^\infty \frac{1}{x^3}\,dx=\frac{1}{2}$

b $\int_2^\infty x^{-\frac{3}{2}}\,dx$

$= \lim_{t\to\infty}\int_2^t x^{-\frac{3}{2}}\,dx$

$= \lim_{t\to\infty}\left[-2x^{-\frac{1}{2}}\right]_2^t$

$= \lim_{t\to\infty}\left(-2t^{-\frac{1}{2}}+\sqrt{2}\right)=\sqrt{2}$

So $\int_2^\infty x^{-\frac{3}{2}}\,dx$ converges and $\int_2^\infty x^{-\frac{3}{2}}\,dx=\sqrt{2}$

c $\int_0^\infty e^{-3x}\,dx$

$= \lim_{t\to\infty}\int_0^t e^{-3x}\,dx$

$= \lim_{t\to\infty}\left[-\frac{e^{-3x}}{3}\right]_0^t$

$= \lim_{t\to\infty}\left(-\frac{e^{-3t}}{3}+\frac{1}{3}\right)=\frac{1}{3}$

So $\int_0^\infty e^{-3x}\,dx$ converges and $\int_0^\infty e^{-3x}\,dx=\frac{1}{3}$

2 a $\int_0^\infty e^x\,dx$

$= \lim_{t\to\infty}\int_0^t e^x\,dx$

$= \lim_{t\to\infty}\left[e^x\right]_0^t$

$= \lim_{t\to\infty}\left(e^t-1\right)$

$e^t\to\infty$ as $t\to\infty$, so the integral diverges.

b $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to\infty}\left[2\sqrt{x}\right]_1^t$

$= \lim_{t\to\infty}\left(2\sqrt{t}-2\right)$

$\sqrt{t}\to\infty$ as $t\to\infty$, so the integral diverges.

c $\int_0^\infty \frac{8x}{\sqrt{1+x^2}}\,dx$

$= \lim_{t\to\infty}\int_0^t \frac{8x}{\sqrt{1+x^2}}\,dx$

$= \lim_{t\to\infty}\left[8\sqrt{1+x^2}\right]_0^t$

$= \lim_{t\to\infty}\left(8\sqrt{1+t^2}-8\right)$

$\sqrt{1+t^2}\to\infty$ as $t\to\infty$, so the integral diverges.

3 a $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to 0}\int_t^1 \frac{1}{\sqrt{x}}\,dx$

$= \lim_{t\to 0}\left[2\sqrt{x}\right]_t^1$

$= \lim_{t\to 0}\left(2-2\sqrt{t}\right)=2$

So $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges and $\int_0^1 \frac{1}{\sqrt{x}}\,dx=2$

3 b $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx$

$= \lim_{t\to\frac{2}{3}} \int_0^t \frac{1}{\sqrt{2-3x}}\,dx$

$= \lim_{t\to\frac{2}{3}} \left[-\frac{2}{3}\sqrt{2-3x}\right]_0^t$

$= \lim_{t\to\frac{2}{3}} \left(-\frac{2}{3}\sqrt{2-3t} + \frac{2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}}{3}$

So $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx$ converges and

$\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}}\,dx = \frac{2\sqrt{2}}{3}$

c $\int_0^{\ln 3} \frac{e^x}{\sqrt{e^x-1}}\,dx$

Let $u = e^x - 1$ and $du = e^x dx$

$\int_0^{\ln 3} \frac{e^x}{\sqrt{e^x-1}}\,dx = \int_0^2 \frac{1}{\sqrt{u}}\,du$

$= \left[2\sqrt{u}\right]_0^2 = 2\sqrt{2}$

4 a $\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$

Split $\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$ as $\int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx + \int_{0}^{2} \frac{1}{\sqrt{|x|}}\,dx$

To find $\int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx$ consider $\int_{-1}^{t} \frac{1}{\sqrt{|x|}}\,dx$

$\int_{-1}^{t} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{t} \frac{1}{\sqrt{-x}}\,dx$

$= \left[-2\sqrt{-x}\right]_{-1}^{t} = -2\sqrt{-t} + 2$

So, $\lim_{t\to 0} \int_{-1}^{t} \frac{1}{\sqrt{|x|}}\,dx = \lim_{t\to 0}\left(-2\sqrt{-t}+2\right) = 2$

4 a Similarly, to find $\int_0^2 \frac{1}{\sqrt{|x|}}\,dx$ consider

$\int_t^2 \frac{1}{\sqrt{|x|}}\,dx = \int_t^2 \frac{1}{\sqrt{x}}\,dx$

$= \left[2\sqrt{x}\right]_t^2 = 2\sqrt{2} - 2\sqrt{t}$

So, $\lim_{t\to 0} \int_t^2 \frac{1}{\sqrt{|x|}}\,dx = \lim_{t\to 0}\left(2\sqrt{2} - 2\sqrt{t}\right)$

$= 2\sqrt{2}$

Since both integrals converge, we know that $\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx$ converges, and

$\int_{-1}^{2} \frac{1}{\sqrt{|x|}}\,dx = \int_{-1}^{0} \frac{1}{\sqrt{|x|}}\,dx + \int_{0}^{2} \frac{1}{\sqrt{|x|}}\,dx$

$= 2 + 2\sqrt{2} = 2\left(1+\sqrt{2}\right)$

b $\int_{-1}^{3} \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$

$= \int_{-1}^{3} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

Split $\int_{-1}^{3} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$ as

$\int_{-1}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx + \int_{0}^{3} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

To find $\int_{-1}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$ consider

$\int_{t}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx = \left[-\sqrt{(3-x)(x+1)}\right]_t^0$

$= -3 + \sqrt{(3-t)(t+1)}$

So,

$\lim_{t\to -1} \int_{t}^{0} \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

$= \lim_{t\to -1}\left(-3 + \sqrt{(3-t)(t+1)}\right)$

$= -3$

4 b Similarly, to find $\int_0^3 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

consider $\int_0^t \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$

$$\int_0^t \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx = \left[-\sqrt{(3-x)(x+1)}\right]_0^t$$

$$= -\sqrt{(3-t)(t+1)} + 3$$

So,

$$\lim_{t\to 3}\int_0^t \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx$$

$$= \lim_{t\to -1}\left(-\sqrt{(3-t)(t+1)} + 3\right)$$

$$= 3$$

Since both integrals converge, we know that $\int_{-1}^3 \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$ converges, and

$$\int_{-1}^3 \frac{x-1}{\sqrt{3+2x-x^2}}\,dx$$

$$= \left(\int_{-1}^0 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx + \int_0^3 \frac{x-1}{\sqrt{(3-x)(x+1)}}\,dx\right)$$

$$= -3 + 3 = 0$$

c $\int_0^{\pi} \tan x\,dx$

Split $\int_0^{\pi} \tan x\,dx$ as $\int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx$

To find $\int_0^{\frac{\pi}{2}} \tan x\,dx$ consider $\int_0^t \tan x\,dx$

$$\int_0^t \tan x\,dx = \left[\ln|\sec x|\right]_0^t = \ln|\sec t|$$

So $\lim_{t\to\frac{\pi}{2}}\int_0^t \tan x\,dx = \lim_{t\to\frac{\pi}{2}}\ln|\sec t|$

$\ln|\sec t| \to \infty$ as $t \to \frac{\pi}{2}$. Therefore, the integral diverges.

5 a $\int \frac{1}{(7-3x)^2}\,dx$

Substitute $u = 7-3x$, so $du = -3dx$

$$\int \frac{1}{(7-3x)^2}\,dx = -\frac{1}{3}\int \frac{1}{u^2}\,du$$

$$= \frac{1}{3u} + C = \frac{1}{3(7-3x)} + C$$

b $\int_{-\infty}^2 \frac{1}{(7-3x)^2}\,dx$

Consider $\int_t^2 \frac{1}{(7-3x)^2}\,dx$

$$\int_t^2 \frac{1}{(7-3x)^2}\,dx = \left[\frac{1}{3(7-3x)}\right]_t^2$$

$$= \frac{1}{3} - \frac{1}{3(7-3t)}$$

$$\lim_{t\to-\infty}\int_t^2 \frac{1}{(7-3x)^2}\,dx$$

$$= \lim_{t\to-\infty}\left(\frac{1}{3} - \frac{1}{3(7-3t)}\right) = \frac{1}{3}$$

Therefore, the integral converges, and

$$\int_{-\infty}^2 \frac{1}{(7-3x)^2}\,dx = \frac{1}{3}$$

6 a $\int x^2 e^{x^3}\,dx$

Substitute $u = x^3$, so $du = 3x^2dx$

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c$$

Therefore, $\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

b $\int_{-\infty}^1 x^2 e^{x^3}\,dx = \lim_{t\to-\infty}\int_t^1 x^2 e^{x^3}\,dx$

$$= \lim_{t\to-\infty}\left[\frac{1}{3}e^{x^3}\right]_t^1 = \lim_{t\to-\infty}\left(\frac{1}{3}e - \frac{1}{3}e^{t^3}\right)$$

$$= \frac{e}{3}$$

Therefore, the integral converges, and

$$\int_{-\infty}^1 x^2 e^{x^3}\,dx = \frac{e}{3}$$

7 a $\int \frac{\ln x}{x}\,dx$

Substitute $u = \ln x$, so $du = \frac{1}{x}dx$

$\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C$

Therefore, $\int \frac{\ln x}{x}dx = \frac{(\ln x)^2}{2} + C$

b $\int_1^{\infty} \frac{\ln x}{x}dx = \lim_{t\to\infty}\int_1^{t} \frac{\ln x}{x}dx$

$= \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty}\left(\frac{(\ln t)^2}{2}\right)$

$\ln t \to \infty$ as $t \to \infty$, so the integral diverges.

8 a $\int (\ln x)^2\,dx$

Integrating by parts,

$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x$

Integrating the second term using integration by parts again,

$\int \ln x\,dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - x$

Therefore,

$\int (\ln x)^2\,dx = x\left((\ln x)^2 - 2\ln x + 2\right) + C$

b $\int_0^1 (\ln x)^2\,dx = \lim_{t\to 0}\int_t^1 (\ln x)^2\,dx$

$= \lim_{t\to 0}\left[x\left((\ln x)^2 - 2\ln x + 2\right)\right]_t^1$

$= \lim_{t\to 0}\left(2 - t\left((\ln t)^2 - 2\ln t + 2\right)\right) = 2$

Since $t(\ln t)^2 \to 0$ and $t\ln t \to 0$ as $t \to 0$, the integral converges.

8 c $\int_0^{\infty} (\ln x)^2\,dx = \int_0^1 (\ln x)^2\,dx + \int_1^{\infty} (\ln x)^2\,dx$

We know that the first integral converges.

$\int_1^{\infty} (\ln x)^2\,dx = \lim_{t\to\infty}\int_1^{\infty} (\ln x)^2\,dx$

$= \lim_{t\to\infty}\left[x\left((\ln x)^2 - 2\ln x + 2\right)\right]_1^{\infty}$

$= \lim_{t\to\infty}\left(t\left((\ln t)^2 - 2\ln t + 2\right) - 2\right)$

Since $t(\ln t)^2 \to \infty$ as $t \to \infty$, the integral diverges.

9 $\int_0^2 \frac{6x}{\sqrt[3]{4-x^2}}dx = \lim_{t\to 2}\int_0^t \frac{6x}{\sqrt[3]{4-x^2}}dx$

Consider $\int \frac{6x}{\sqrt[3]{4-x^2}}dx$

Substitute $u = 4 - x^2$, so $du = -2x\,dx$

$\int \frac{6x}{\sqrt[3]{4-x^2}}dx = \int -\frac{3}{\sqrt[3]{u}}du = -\frac{9}{2}u^{\frac{2}{3}} + C$

$\lim_{t\to 2}\int_0^t \frac{6x}{\sqrt[3]{4-x^2}}dx$

$= \lim_{t\to 2}\left[-\frac{9}{2}\sqrt[3]{(4-x^2)^2}\right]_0^t$

$= \lim_{t\to 2}\left(-\frac{9}{2}\sqrt[3]{(4-t^2)^2} + 9\sqrt[3]{2}\right) = 9\sqrt[3]{2}$

10 $\int_{-2}^{2} \frac{\sqrt{2-x} - 3\sqrt{2+x}}{\sqrt{4-x^2}}dx$

$= \int_{-2}^{2} \frac{\sqrt{2-x} - 3\sqrt{2+x}}{\sqrt{(2-x)(2+x)}}dx$

$= \int_{-2}^{2}\left(\frac{1}{\sqrt{2+x}} - \frac{3}{\sqrt{2-x}}\right)dx$

$= \int_{-2}^{2}\frac{1}{\sqrt{2+x}}dx - \int_{-2}^{2}\frac{3}{\sqrt{2-x}}dx$

$\int_{-2}^{2}\frac{1}{\sqrt{2+x}}dx = \lim_{t\to -2}\int_t^2 \frac{1}{\sqrt{2+x}}dx$

$= \lim_{t\to -2}\left[2\sqrt{2+x}\right]_t^2$

$= \lim_{t\to -2}\left(4 - 2\sqrt{2+t}\right) = 4$

10 $\int_{-2}^{2} \frac{3}{\sqrt{2-x}}\,dx = \lim_{t\to 2}\int_{-2}^{t} \frac{1}{\sqrt{2-x}}\,dx$

$= \lim_{t\to 2}\left[-6\sqrt{2-x}\right]_{-2}^{t}$

$= \lim_{t\to 2}\left(-6\sqrt{2-t}+12\right) = 12$

So,

$\int_{-2}^{2} \frac{\sqrt{2-x}-3\sqrt{2+x}}{\sqrt{4-x^2}}\,dx$

$= \int_{-2}^{2} \frac{1}{\sqrt{2+x}}\,dx - \int_{-2}^{2} \frac{3}{\sqrt{2-x}}\,dx$

$= 4 - 12 = -8$

11 The area enclosed by the curve in the figure is given by $\left|\int_0^1 \ln x\right|$ which may be written as

$\left|\lim_{t\to 0}\int_t^1 \ln x\,dx\right|$

Integrating by parts,

$\left|\lim_{t\to 0}\int_t^1 \ln x\,dx\right| = \left|\lim_{t\to 0}\left[x\ln x - x\right]_t^1\right|$

$= \left|\lim_{t\to 0}\left(-1 - t\ln t + t\right)\right| = 1$

12 a Since $\tan x$ is undefined in the upper limit $x = \frac{\pi}{2}$, $\int_0^{\frac{\pi}{2}} \tan x\,dx$ is an improper integral.

b $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Substitute $u = \cos x$, so $du = -\sin x\,dx$

$\int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du$

$= -\ln u + C = \ln\frac{1}{u} + C = \ln(\sec x) + C$

$\int_0^{\frac{\pi}{2}} \tan x\,dx = \lim_{t\to\frac{\pi}{2}}\int_0^t \tan x\,dx$

$= \lim_{t\to\frac{\pi}{2}}\left[-\ln(\cos x)\right]_0^t = \lim_{t\to\frac{\pi}{2}}\left(-\ln(\cos t)\right)$

$-\ln(\cos t) \to \infty$ as $t \to \frac{\pi}{2}$ so the integral diverges.

13 a $\sec^2 x$ is undefined at $x = \frac{\pi}{2}$ so the integral should be split up at that point.

b $\int_0^{\pi} \sec^2 x\,dx = \int_0^{\frac{\pi}{2}} \sec^2 x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sec^2 x\,dx$

$\int_0^{\frac{\pi}{2}} \sec^2 x\,dx = \lim_{t\to\frac{\pi}{2}}\int_0^t \sec^2 x\,dx$

$= \lim_{t\to\frac{\pi}{2}}\left[\tan x\right]_0^t = \lim_{t\to\frac{\pi}{2}}\tan t$

$\tan t \to \infty$ as $t \to \frac{\pi}{2}$ so the integral diverges.

14 $\int_1^{\infty} \frac{1}{x^a}\,dx$

If $a = 1$,

$\int_1^{\infty} \frac{1}{x^a}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty}\left[\ln x\right]_1^t = \lim_{t\to\infty}\ln t$

$\ln t \to \infty$ as $t \to \infty$ so the integral diverges.

If $a \neq 1$,

$\int_1^{\infty} \frac{1}{x^a}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^a}\,dx$

$= \lim_{t\to\infty}\left[\frac{x^{-a+1}}{-a+1}\right]_1^t$

$= \lim_{t\to\infty}\left(\frac{t^{-a+1}}{-a+1} - \frac{1}{-a+1}\right)$

If $a < 1$, $\frac{t^{-a+1}}{-a+1} \to \infty$ as $t \to \infty$ so the integral diverges.

If $a > 1$, $\frac{t^{-a+1}}{-a+1} \to 0$ as $t \to \infty$ so the integral converges, and

$\lim_{t\to\infty}\left(\frac{t^{-a+1}}{-a+1} - \frac{1}{-a+1}\right) = \frac{1}{a-1}$

Therefore, $a > 1$, and $\int_1^{\infty} \frac{1}{x^a}\,dx = \frac{1}{a-1}$

15 a $\displaystyle\int_0^k \frac{1}{2x^2+3x+1}\,dx = \int_0^k \frac{1}{(x+1)(2x+1)}\,dx$

$$= \int_0^k \left(\frac{2}{2x+1} - \frac{1}{x+1}\right) dx$$

$$= \left[\ln(2x+1) - \ln(x+1)\right]_0^k$$

$$= \left[\ln\left(\frac{2x+1}{x+1}\right)\right]_0^k$$

$$= \ln\left(\frac{2k+1}{k+1}\right)$$

b $\displaystyle\int_0^\infty \frac{1}{2x^2+3x+1}\,dx = \lim_{k\to\infty} \ln\left(\frac{2k+1}{k+1}\right)$

$$= \lim_{k\to\infty} \ln\left(\frac{2+\frac{1}{k}}{1+\frac{1}{k}}\right) = \ln 2$$

Challenge

$\int e^{-x}\sin^2 x\,dx$

Using trigonometric identities,

$$\int e^{-x}\sin^2 x\,dx = \int e^{-x}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= -\frac{1}{2}e^{-x} - \frac{1}{2}\int e^{-x}\cos 2x\,dx$$

Integrating by parts,

$$\int e^{-x}\cos 2x\,dx = \left(\begin{array}{l} -e^{-x}\cos 2x - \\ 2\int e^{-x}\sin 2x\,dx \end{array}\right)$$

Integrating by parts again,

$$\int e^{-x}\sin 2x\,dx = \left(\begin{array}{l} -e^{-x}\sin 2x + \\ 2\int e^{-x}\cos 2x\,dx \end{array}\right)$$

Adding the above two equations gives,

$$\int e^{-x}\cos 2x\,dx = \left(\begin{array}{l} -e^{-x}\cos 2x + \\ 2e^{-x}\sin 2x - \\ 4\int e^{-x}\cos 2x\,dx \end{array}\right)$$

$$5\int e^{-x}\cos 2x\,dx = -e^{-x}\cos 2x + 2e^{-x}\sin 2x$$

$$\int e^{-x}\cos 2x\,dx = \frac{1}{5}\left(\begin{array}{l} -e^{-x}\cos 2x + \\ 2e^{-x}\sin 2x \end{array}\right)$$

Therefore,

$\int e^{-x}\sin^2 x\,dx$

$$= -\frac{1}{2}e^{-x} - \frac{1}{10}\left(-e^{-x}\cos 2x + 2e^{-x}\sin 2x\right)$$

$$= \frac{1}{10}e^{-x}\left(-2\sin 2x + \cos 2x - 5\right)$$

So,

$$\int_0^\infty e^{-x}\sin^2 x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin^2 x\,dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{10}e^{-x}\left(-2\sin 2x + \cos 2x - 5\right)\right]_0^t$$

$$= \lim_{t\to\infty}\left(\frac{1}{10}e^{-t}\left(-2\sin 2t + \cos 2t - 5\right) - \left(-\frac{2}{5}\right)\right)$$

$e^{-t}\sin 2t \to 0$ and $e^{-t}\cos 2t \to 0$ as $t\to\infty$.

Therefore,

$$\int_0^\infty e^{-x}\sin^2 x\,dx = \frac{2}{5}$$