## Series 2C

**1 a** $f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$

$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2} \Rightarrow f'(0) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$

$f'''(x) = -3.2(1-x)^{-4}(-1) = 3.2(1-x)^{-4} \Rightarrow f'''(0) = 3!$

General term: The pattern here is such that $f^{(r)}(x)$ can be written down

$$f^{(r)}(x) = r(r-1)\ldots 2(1-x)^{-(r+1)} = r!(1-x)^{-(r+1)} \Rightarrow f^{(r)}(0) = r!$$

Using $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$

$$(1-x)^{-1} = 1 + x + \frac{2}{2!}x^2 + \ldots + \frac{r!}{r!}x^r + \ldots = 1 + x + x^2 + \ldots + x^r + \ldots$$

**b** $f(x) = \sqrt{(1+x)} = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

Using Maclaurin's expansion

$$\sqrt{(1+x)} = 1 + \frac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{8}\right)}{3!}x^3 - \ldots$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$$

**2** $f(x) = e^{\sin x} \Rightarrow f(0) = 1$

$f'(x) = \cos x\, e^{\sin x} \Rightarrow f'(0) = 1$

$f''(x) = \cos^2 x\, e^{\sin x} - \sin x\, e^{\sin x} \Rightarrow f''(0) = 1$

Substituting into Maclaurin's expansion gives

$$e^{\sin x} = 1 + 1x + \frac{1}{2!}x^2 + \ldots$$
$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

**3 a** $f(x) = \cos x \quad \Rightarrow f(0) = 1$

$f'(x) = -\sin x \quad \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \quad \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \quad \Rightarrow f'''(0) = 0$

$f''''(x) = \cos x \quad \Rightarrow f''''(0) = 1$

The process repeats itself after every 4th derivative, like sin $x$ does.
Using Maclaurin's expansion, only even powers of $x$ are produced.

$$\cos x = 1 + \frac{(-1)}{2!}x^2 + \frac{1}{4!}x^4 + \ldots + \frac{(-1)^r}{(2r)!}x^{2r} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$$

**b** Using $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ with $x = \frac{\pi}{6}$ (must be in radians)

$\cos \approx 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31\,104} = 0.86605\ldots$ which is correct to 3 d.p.

**4 a** Substituting $x = 1$ into the Maclaurin expansion of $e^x$, gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \ldots$$

The approximations, to 4 d.p. where necessary, using $n$ terms of the series are

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Approx. | 1 | 2 | 2.5 | 2.6667 | 2.7083 | 2.7167 | 2.7181 | 2.7183 | 2.7183 | 2.7183 |

So $e = 2.718$ (3 d.p.)

**b** Substituting $x = 0.2$ into the Maclaurin expansion of $\ln(1+x)$, gives

$$\ln\left(\frac{6}{5}\right) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6} + \frac{(0.2)^7}{7} - \ldots$$

The approximations, to 4 d.p. where necessary, using $n$ terms of the series are

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Approximation | 0.2 | 0.18 | 0.1827 | 0.1823 | 0.1823 |

So $\ln\left(\frac{6}{5}\right) = 0.182$ (3 d.p.)

**5 a** $f(x)=e^{3x}, f^{(n)}(x)=3^n e^{3x}$

So $f(0)=1,\ f'(0)=3,\ f''(0)=3^2,\ f'''(0)=3^3,\ f''''(0)=3^4$

$$f(x)=e^{3x}=1+3x+\frac{3^2}{2!}x^2+\frac{3^3}{3!}x^3+\frac{3^4}{4!}x^4+\ldots$$

$$=1+3x+\frac{9x^2}{2}+\frac{9x^3}{2}+\frac{27}{8}x^4+\ldots\left[\text{Note: this is }1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\frac{(3x)^4}{4!}+\ldots\right]$$

**b** As $f(x)=\ln(1+2x)$, $\quad f(0)=\ln 1=0$

$f'(x)=\frac{2}{1+2x}=2(1+2x)^{-1}$, $\quad f'(0)=2$

$f''(x)=-4(1+2x)^{-2}$, $\quad f''(0)=-4$

$f'''(x)=16(1+2x)^{-3}$, $\quad f'''(0)=16$

$f''''(x)=-96(1+2x)^{-4}$, $\quad f''''(0)=-96$

$$\text{So }\ln(1+2x)=0+2x+\frac{(-4)}{2!}x^2+\frac{(16)}{3!}x^3+\frac{(-96)}{4!}x^4+\ldots$$

$$=2x-2x^2+\frac{8x^3}{3}-4x^4+\ldots\left[\text{Note: this is }2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-\frac{(2x)^4}{4}+\ldots\right]$$

**c** $f(x)=\sin^2 x$ $\quad f(0)=0$

$f'(x)=2\sin x\cos x=\sin 2x$ $\quad f'(0)=0$

$f''(x)=2\cos 2x$ $\quad f''(0)=2$

$f'''(x)=-4\sin 2x$ $\quad f'''(0)=0$

$f''''(x)=-8\cos 2x$ $\quad f''''(0)=-8$

$$\text{So, }f(x)=\sin^2 x=0+0x+\frac{2}{2!}x^2+0x^3+\frac{(-8)}{4!}x^4+\ldots=x^2-\frac{x^4}{3}+\ldots$$

**6** $$\cos\left(x-\frac{\pi}{4}\right)=\cos x\cos\left(\frac{\pi}{4}\right)+\sin x\sin\left(\frac{\pi}{4}\right)$$

Use $\cos(A-B)=\cos A\cos B+\sin A\sin B$.

$$=\frac{1}{\sqrt{2}}(\cos x+\sin x)$$

$$=\frac{1}{\sqrt{2}}\left\{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)+\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)\right\}$$

$$=\frac{1}{\sqrt{2}}\left(1+x-\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\ldots\right)$$

**7 a** $f(x)=(1-x)^2\ln(1-x)$

$$f'(x)=(1-x^2)\times\frac{(-1)}{1-x}+2(1-x)(-1)\ln(1-x)$$

Use the product rule.

$$=x-1-2(1-x)\ln(1-x)$$

$$f''(x)=1-2\left[(1-x)\times\frac{(-1)}{1-x}-\ln(1-x)\right]=1+2+2\ln(1-x)=3+2\ln(1-x)$$

**7** **b** $f'''(x) = \dfrac{-2}{1-x}$

Substituting $x = 0$ in all the results gives

$f(0) = 0,\ f'(0) = -1,\ f''(0) = 3,\ f'''(0) = -2$

**c** $f(x) = (1-x)^2 \ln(1-x) = 0 + (-1)x + \dfrac{3}{2!}x^2 + \dfrac{(-2)}{3!}x^3 + \ldots$

$$= -x + \frac{3x^2}{2} - \frac{1}{3}x^3$$

**8** **a** Using the series expansions for $\sin x$ and $\cos x$ as far as the term in $x^5$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$$

so $3\sin x - 4x\cos x + x = 3\left(x - \dfrac{1}{6}x^3 + \dfrac{1}{120}x^5 - \ldots\right) - 4x\left(1 - \dfrac{1}{2}x^2 + \dfrac{1}{24}x^4 - \ldots\right) + x$

$$= 3x - \frac{1}{2}x^3 + \frac{1}{40}x^5 - 4x + 2x^3 - \frac{1}{6}x^5 + x + \ldots$$

$$3\sin x - 4x\cos x + x = \frac{3}{2}x^3 - \frac{17}{120}x^5 + \ldots$$

**b** $\dfrac{3\sin x - 4x\cos x + x}{x^3} = \dfrac{3}{2} - \dfrac{17}{120}x^2$ + higher powers in $x$ using **a**

Hence, the limit, as $x \to 0$, is $\dfrac{3}{2}$.

**9** **a** $f(x) = \ln\cos x \qquad \Rightarrow f(0) = 0$

$f'(x) = \dfrac{1}{\cos x} \times (-\sin x) \qquad \Rightarrow f'(0) = 0$

$= -\tan x$

**b** $f''(x) = -\sec^2 x \qquad \Rightarrow f''(0) - 1$

$f'''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x \qquad \Rightarrow f'''(0) = 0$

$f''''(x) = -2\{\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)\} \qquad \Rightarrow f''''(0) = -2$

**c** Substituting into Maclaurin's expansion

$$\ln\cos x = 0 + 0x + \frac{(-1)}{2!}x^2 + 0x^3 + \frac{(-2)}{4!}x^4 + \ldots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$$

**9 d** Substituting $x = \frac{\pi}{4}$ gives $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{1}{12}\left(\frac{\pi^4}{256}\right)$

but but $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2,$

so $-\frac{1}{2}\ln 2 = -\frac{\pi^2}{2.16} - \frac{\pi^4}{12.256} + \ldots$

$\Rightarrow \ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{6.256}$, using only first two terms.

$= \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

**10**

$f(x) = \tan x \quad \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x \quad \Rightarrow f'(0) = 1$

$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x \quad \Rightarrow f''(0) = 0$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \quad \Rightarrow f'''(0) = 2$

$f''''(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x \quad \Rightarrow f''''(0) = 0$ as $\tan(0) = 0$

$f'''''(x) = 16\sec^2 x \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x \quad \Rightarrow f'''''(0) = 16$ as $\tan(0) = 0$ and $\sec(0) = 1$

Substituting into Maclaurin's expansion gives

$\tan x = 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^3 + \frac{16}{5!}x^5 + \ldots$

$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

**Challenge**

**a** $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0, \forall x \in \mathbb{R}$

so the Maclaurin expansion of

$e^x$ converges for all $x \in \mathbb{R}$

**b** $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$

$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n x^{n+1}}{n+1}}{\frac{(-1)^{n-1}x^n}{n}}\right| = \lim_{n\to\infty}\left|-x\frac{n}{n+1}\right| = |x|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |x|$

so the Maclaurin expansion of

$\ln(1+x)$ converges for $|x| < 1$

and diverges for $|x| > 1$