

Series 2B

1

	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(n)}(x)$
a	$2e^{2x}$	$2^2 e^{2x} = 4e^{2x}$	$2^3 e^{2x} = 8e^{2x}$	$2^n e^{2x}$
b	$n(1+x)^{n-1}$	$n(n-1)(1+x)^{n-2}$	$n(n-1)(n-2)(1+x)^{n-3}$	$n!$
c	$e^x + xe^x$	$e^x + (e^x + xe^x) = 2e^x + xe^x$	$2e^x + (e^x + xe^x) = 3e^x + xe^x$	$ne^x + xe^x$
d	$(1+x)^{-1}$	$-(1+x)^{-2}$	$(-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$	$(-1)^{n-1}(n-1)!(1+x)^{-n}$

2 a $y = e^{2+3x}$, so $\frac{dy}{dx} = 3e^{2+3x}$, $\frac{d^2y}{dx^2} = 3^2 e^{2+3x}$, $\frac{d^3y}{dx^3} = 3^3 e^{2+3x}$, and so on.

It follows that $\frac{d^n y}{dx^n} = 3^n e^{2+3x} = 3^n y$ as $y = e^{2+3x}$.

b

$$y = e^{2+3x}$$

$$y' = 3e^{2+3x} = 3y$$

$$y'' = 3y' = 3^2 y$$

$$\frac{d^n y}{dx^n} = 3^n y$$

When $x = \log \frac{1}{9}$, $\frac{d^6 y}{dx^6} = 3^6 e^{2+3 \log \frac{1}{9}}$

$$= 3^6 e^2 \left(\frac{1}{9}\right)^3 = e^2$$

3 a $y = \sin^2 3x = (\sin 3x)^2$, so $\frac{dy}{dx} = 2(\sin 3x)(3 \cos 3x)$
 $= 3(2 \sin 3x \cos 3x)$
 $= 3 \sin 6x$

Use $\frac{du^n}{dx} = nu^{n-1} \frac{du}{dx}$.

Use $\sin 2A = 2 \sin A \cos A$.

b $\frac{d^2 y}{dx^2} = 18 \cos 6x$, $\frac{d^3 y}{dx^3} = -108 \sin 6x$, $\frac{d^4 y}{dx^4} = -648 \cos 6x$

c $y'''' = -648 \cos 6x$

When $x = \frac{\pi}{6}$, $\frac{d^4 y}{dx^4} = -648 \cos \pi = 648$

4 a $f'(x) = 2xe^{-x} - x^2 e^{-x}$

$$f''(x) = (2e^{-x} - 2xe^{-x}) - (2xe^{-x} - x^2 e^{-x}) = e^{-x}(2 - 4x + x^2)$$

$$f'''(x) = e^{-x}(-4 + 2x) - e^{-x}(2 - 4x + x^2) = e^{-x}(-6 + 6x - x^2)$$

b $f'''(x) = e^{-x}(6 - 2x) - e^{-x}(-6 + 6x - x^2) = e^{-x}(12 - 8x + x^2)$

so $f'''(2) = e^{-2}(12 - 16 + 4) = 0$

5 a Given that $y = \sec x$, so $\frac{dy}{dx} = \sec x \tan x$

$$\frac{d^2y}{dx^2} = \sec x(\sec^2 x) + (\sec x \tan x) \tan x$$

Use the product rule.

$$= \sec x(\sec^2 x + \tan^2 x)$$

$$= \sec x(\sec^2 x + \sec^2 x - 1)$$

Use $1 + \tan^2 A = \sec^2 A$.

$$= 2\sec^3 x - \sec x$$

b $y'' = 2y^3 - y$

$$y''' = 6y^2 y' - y'$$

$$= (6\sec^2 x - 1) \tan x \sec x$$

$$\text{When } x = \frac{\pi}{4}, y''' = (6\sec^2 \frac{\pi}{4} - 1) \tan \frac{\pi}{4} \sec \frac{\pi}{4}$$

$$= (6(\sqrt{2})^2 - 1) \times 1 \times \sqrt{2} = 11\sqrt{2}$$

6 a $\frac{d}{dx}(y^2) = \frac{d}{dx}(y^2) \frac{dy}{dx} = 2y \frac{dy}{dx}$

Use the chain rule.

$$\frac{d^2}{dx^2}(y^2) = \frac{d}{dx} \left(2y \frac{dy}{dx} \right) = 2y \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \frac{dy}{dx} = 2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2$$

Use the product rule.

b $\frac{d^3}{dx^3}(y^2) = \frac{d}{dx} \left(2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 \right)$

$$= 2 \left\{ y \frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} \times \frac{d^2y}{dx^2} \right\}$$

$$= 2 \left\{ y \frac{d^3y}{dx^3} + 3 \frac{dy}{dx} \times \frac{d^2y}{dx^2} \right\}$$

$$7 \quad f(x) = \ln \left\{ x + \sqrt{1+x^2} \right\}$$

$$\begin{aligned} \mathbf{a} \quad f'(x) &= \frac{1}{x + \sqrt{1+x^2}} \times \left\{ 1 + \frac{x}{\sqrt{1+x^2}} \right\}, \\ &= \frac{1}{\cancel{x + \sqrt{1+x^2}}} \times \left\{ \frac{\cancel{\sqrt{1+x^2}} + x}{\sqrt{1+x^2}} \right\} = \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Use $\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx}$.

$$\text{So } \sqrt{1+x^2} f'(x) = 1$$

b Differentiating this equation w.r.t. x , using the product rule

$$\sqrt{1+x^2} f''(x) + \frac{x}{\sqrt{1+x^2}} f'(x) = 0$$

$$\text{So } (1+x^2)f''(x) + xf'(x) = 0$$

Multiply through by $\sqrt{1+x^2}$.

c Differentiating this results w.r.t. x

$$\{(1+x^2)f'''(x) + 2xf''(x)\} + \{f'(x) + xf''(x)\} = 0$$

giving

$$(1+x^2)f'''(x) + 3xf''(x) + f'(x) = 0$$

$$\mathbf{d} \quad f'(0) = \frac{1}{\sqrt{1+0}} = 1$$

Using $(1+x^2)f''(x) + xf'(x) = 0$ with $x = 0$ and $f'(0) = 1$

$$f''(0) + (0)(1) = 0 \Rightarrow f''(0) = 0$$

Using $(1+x^2)f'''(x) + 3xf''(x) + f'(x) = 0$ with $x = 0$, $f'(0) = 1$ and $f''(0) = 0$

$$f'''(0) + (0)(0) + 1 = 0 \Rightarrow f'''(0) = -1$$