## Complex numbers 1G

**1** **a** One vertex corresponds to the complex number $z = 4i = 4e^{\frac{\pi i}{2}}$, so $z = (0,4)$. Let $\omega = e^{\frac{2\pi i}{3}}$ then the other two correspond to

$$z\omega = 4e^{\frac{\pi i}{2}} \times e^{\frac{2\pi i}{3}} = 4e^{\frac{7\pi i}{6}} = 4\cos\frac{7\pi}{6} + 4i\sin\frac{7\pi}{6} = -2\sqrt{3} - 2i = \left(-2\sqrt{3}, -2\right)$$

and

$$z\omega^2 = 4e^{\frac{\pi i}{2}} \times e^{\frac{4\pi i}{3}} = 4e^{\frac{11\pi i}{6}} = 4\cos\frac{11\pi}{6} + 4i\sin\frac{11\pi}{6} = 2\sqrt{3} - 2i = \left(2\sqrt{3}, -2\right)$$

**b** We are given one vertex corresponds to $z = (5,0)$. Let $\omega = e^{\frac{\pi i}{2}}$ be a primitive $4^{\text{th}}$ root of unity then the other three vertices are given by

$$z\omega = 5 \times e^{\frac{\pi i}{2}} = 5i = (0,5)$$

$$z\omega^2 = 5 \times e^{i\pi} = -5 = (-5,0)$$

$$z\omega^3 = 5 \times e^{\frac{3\pi i}{2}} = -5i = (0,-5)$$

**c** We are given that one vertex corresponds to $z = -1 + i\sqrt{3} = 2e^{\frac{2\pi i}{3}}$ which has the coordinate $(-1, \sqrt{3})$. Let $\omega = e^{\frac{2\pi i}{5}}$ be a primitive $5^{\text{th}}$ root of unity then the other four vertices are given by

$$z\omega = 2e^{\frac{2\pi i}{3}} \times e^{\frac{2\pi i}{5}} = 2e^{\frac{16\pi i}{15}} = 2\cos\frac{16\pi}{15} + 2i\sin\frac{16\pi}{15} = \left(2\cos\frac{16\pi}{15}, 2\sin\frac{16\pi}{15}\right)$$

$$z\omega^2 = 2e^{\frac{2\pi i}{3}} \times e^{\frac{4\pi i}{5}} = 2e^{\frac{22\pi i}{15}} = 2\cos\frac{22\pi}{15} + 2i\sin\frac{22\pi}{15} = \left(2\cos\frac{22\pi}{15}, 2\sin\frac{22\pi}{15}\right)$$

$$z\omega^3 = 2e^{\frac{2\pi i}{3}} \times e^{\frac{6\pi i}{5}} = 2e^{\frac{28\pi i}{15}} = 2\cos\frac{28\pi}{15} + 2i\sin\frac{28\pi}{15} = \left(2\cos\frac{28\pi}{15}, 2\sin\frac{28\pi}{15}\right)$$

$$z\omega^4 = 2e^{\frac{2\pi i}{3}} \times e^{\frac{8\pi i}{5}} = 2e^{\frac{4\pi i}{15}} = 2\cos\frac{4\pi}{15} + 2i\sin\frac{4\pi}{15} = \left(2\cos\frac{4\pi}{15}, 2\sin\frac{4\pi}{15}\right)$$

**d** We are given that one vertex corresponds to $z = 2 + 2i = 2\sqrt{2}e^{\frac{\pi i}{4}}$ which has the coordinate $(2,2)$

Let $\omega = e^{\frac{\pi i}{3}}$ be a primitive $6^{\text{th}}$ root of unity then the other five vertices of the hexagon are given by

$$z\omega = 2\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{\pi i}{3}} = 2\sqrt{2}e^{\frac{7\pi i}{12}} = 2\sqrt{2}\cos\frac{7\pi}{12} + 2i\sqrt{2}\sin\frac{7\pi}{12} = \left(2\sqrt{2}\cos\frac{7\pi}{12}, 2\sqrt{2}\sin\frac{7\pi}{12}\right)$$

$$z\omega^2 = 2\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{2\pi i}{3}} = 2\sqrt{2}e^{\frac{11\pi i}{12}} = 2\sqrt{2}\cos\frac{11\pi}{12} + 2i\sqrt{2}\sin\frac{11\pi}{12} = \left(2\sqrt{2}\cos\frac{11\pi}{12}, 2\sqrt{2}\sin\frac{11\pi}{12}\right)$$

$$z\omega^3 = 2\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{3\pi i}{3}} = 2\sqrt{2}e^{\frac{15\pi i}{12}} = 2\sqrt{2}\cos\frac{15\pi}{12} + 2i\sqrt{2}\sin\frac{15\pi}{12} = \left(2\sqrt{2}\cos\frac{15\pi}{12}, 2\sqrt{2}\sin\frac{15\pi}{12}\right)$$

$$z\omega^4 = 2\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{4\pi i}{3}} = 2\sqrt{2}e^{\frac{19\pi i}{12}} = 2\sqrt{2}\cos\frac{19\pi}{12} + 2i\sqrt{2}\sin\frac{19\pi}{12} = \left(2\sqrt{2}\cos\frac{19\pi}{12}, 2\sqrt{2}\sin\frac{19\pi}{12}\right)$$

$$z\omega^5 = 2\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{5\pi i}{3}} = 2\sqrt{2}e^{\frac{23\pi i}{12}} = 2\sqrt{2}\cos\frac{23\pi}{12} + 2i\sqrt{2}\sin\frac{23\pi}{12} = \left(2\sqrt{2}\cos\frac{23\pi}{12}, 2\sqrt{2}\sin\frac{23\pi}{12}\right)$$

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**2** First we translate so that the centre of the triangle is the origin, this maps the vertex at $(3,-2)=3-2i$ gets mapped to $z=3-2i-(2+3i)=1-5i$ since the centre of the translated triangle now lies at the origin we can find the other two vertices of the translated triangle by multiplying by a primitive 3rd root of unity $\omega=e^{\frac{2\pi i}{3}}$, this gives

$$z\omega=(1-5i)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{5\sqrt{3}-1}{2}+\left(\frac{\sqrt{3}+5}{2}\right)i$$

$$z\omega^2=(1-5i)\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\frac{-5\sqrt{3}-1}{2}+\left(\frac{-\sqrt{3}+5}{2}\right)i$$

Hence when we reverse the translation to take the centre of the triangle is at $(2,3)$ gives the other two vertices of the triangle at

$$z_2=\frac{5\sqrt{3}-1}{2}+\left(\frac{\sqrt{3}+5}{2}\right)i+2+3i=\frac{5\sqrt{3}+3}{2}+\left(\frac{\sqrt{3}+11}{2}\right)i$$

$$z_3=\frac{-5\sqrt{3}-1}{2}+\left(\frac{-\sqrt{3}+5}{2}\right)i+2+3i=\frac{-5\sqrt{3}+3}{2}+\left(\frac{-\sqrt{3}+11}{2}\right)i$$

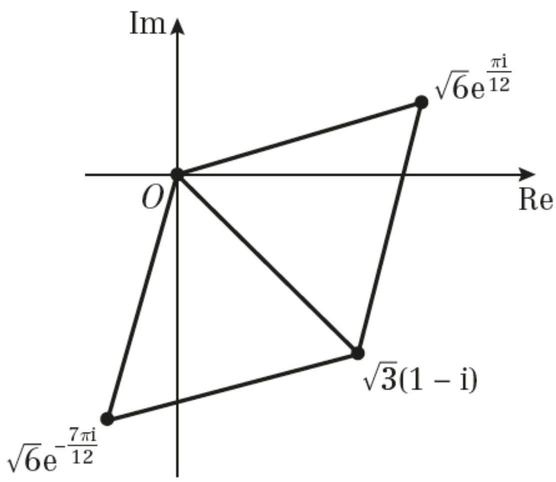
So the coordinates are $z_2=\left(\frac{5\sqrt{3}+3}{2},\frac{\sqrt{3}+11}{2}\right), z_3=\left(\frac{-5\sqrt{3}+3}{2},\frac{-\sqrt{3}+11}{2}\right)$

**3** We have that $A$ corresponds to the complex number $z=\sqrt{3}(1-i)=\sqrt{6}e^{-\frac{\pi i}{4}}$, then $B$ corresponds to rotating $A$ by $\pm\frac{\pi}{3}$ hence the complex number $b$ representing $B$ is one of the two possible values

$$b=\sqrt{6}e^{-\frac{\pi i}{4}}\times e^{\frac{\pi i}{3}}=\sqrt{6}e^{\frac{\pi i}{12}}$$

$$b=\sqrt{6}e^{-\frac{\pi i}{4}}\times e^{-\frac{\pi i}{3}}=\sqrt{6}e^{-\frac{7\pi i}{12}}$$

$\sqrt{6}\,e^{\frac{\pi i}{12}}, \sqrt{6}\,e^{-\frac{7\pi i}{12}}$

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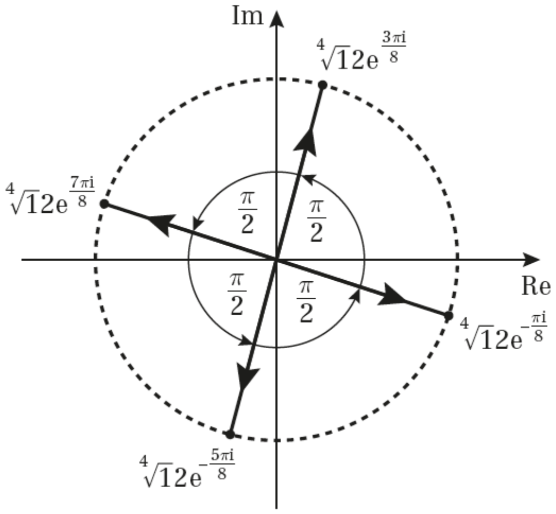
**4 a** We have $z = -12i = 12e^{-\frac{\pi i}{2}}$ so if $\omega = re^{i\theta}$ satisfies $\omega^4 = z$ then we have $r = \sqrt[4]{12}$ and $4\theta = -\frac{\pi}{2} + 2k\pi$ for some $k \in \mathbb{Z}$ hence 4 distinct roots are

$\omega_1 = \sqrt[4]{12}e^{-\frac{\pi i}{8}}$

$\omega_2 = \sqrt[4]{12}e^{\frac{3\pi i}{8}}$

$\omega_3 = \sqrt[4]{12}e^{\frac{7\pi i}{8}}$

$\omega_4 = \sqrt[4]{12}e^{-\frac{5\pi i}{8}}$




**b** Let the points representing these roots in order of increasing $\theta$ be $A, B, C, D$ which form a square in the complex plane then by geometrical considerations the angle between adjacent midpoints of edges is $\frac{\pi}{2}$ and each midpoint has the same modulus. Hence all 4 midpoints are 4th roots of the same complex number $w$ and to find $w$ it suffices to compute the midpoint of one edge and take the fourth power, so without loss of generality we compute the midpoint of $\overrightarrow{AB}$ this is given by

$\frac{1}{2}\left(\sqrt[4]{12}e^{-\frac{5\pi i}{8}} + \sqrt[4]{12}e^{-\frac{\pi i}{8}}\right)$

Geometrically the argument is $\frac{1}{2}\left(-\frac{5\pi i}{8} + -\frac{\pi i}{8}\right) = -\frac{3\pi}{8}$ and by considering a triangle with vertices

At the origin, $\omega_4$ and $\omega_1$ the modulus is $\sqrt[4]{12}\sin\frac{\pi}{4} = \frac{\sqrt[4]{12}}{\sqrt{2}} = \sqrt[4]{3}$ hence the midpoint is given by

$\sqrt[4]{3}e^{-\frac{3\pi i}{8}}$ and so $w = \left(\sqrt[4]{3}e^{-\frac{3\pi i}{8}}\right)^4 = 3e^{-\frac{12\pi i}{8}} = 3e^{\frac{\pi i}{2}} = 3i$

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**5 a** Let $z = 8 + 8i = 8\sqrt{2}e^{\frac{\pi i}{4}}$ and $\omega = e^{\frac{\pi i}{3}}$ be a primitive 6th root of unity then the other 5 vertices are given by

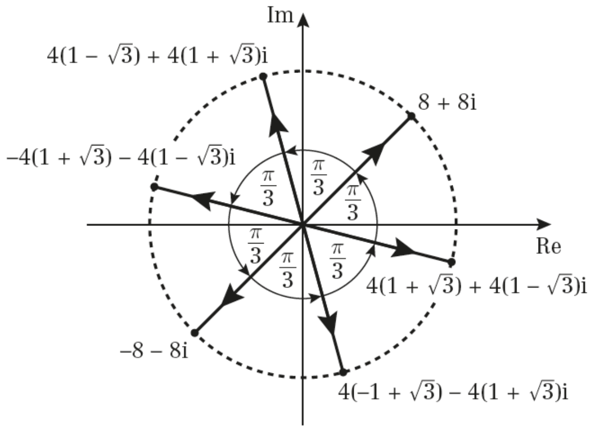
$$z\omega = 8\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{\pi i}{3}} = 8\sqrt{2}e^{\frac{7\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) = 4\left(1-\sqrt{3}\right) + 4i\left(1+\sqrt{3}\right)$$

$$z\omega^2 = 8\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{2\pi i}{3}} = 8\sqrt{2}e^{\frac{11\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) = -4\left(1+\sqrt{3}\right) + 4i\left(-1+\sqrt{3}\right)$$

$$z\omega^3 = 8\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{3\pi i}{3}} = 8\sqrt{2}e^{\frac{5\pi i}{4}} = 8\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -8 - 8i$$

$$z\omega^4 = 8\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{4\pi i}{3}} = 8\sqrt{2}e^{\frac{19\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) = 4\left(-1+\sqrt{3}\right) - 4i\left(1+\sqrt{3}\right)$$

$$z\omega^5 = 8\sqrt{2}e^{\frac{\pi i}{4}} \times e^{\frac{5\pi i}{3}} = 8\sqrt{2}e^{\frac{23\pi i}{12}} = 8\sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) = 4\left(1+\sqrt{3}\right) + 4i\left(1-\sqrt{3}\right)$$




**b** When we square the vertices, the figure becomes an equilateral triangle since vertices that differ in argument by $\pi$ get squared to the same value hence to find the vertices of the triangle we only need to square the first 3 vertices listed above which gives

$$z^2 = \left(8+8i\right)^2 = 128e^{\frac{\pi i}{2}} = 128i$$

$$\left(z\omega\right)^2 = 128e^{\frac{7\pi i}{6}} = 128\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 128\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$\left(z\omega^2\right) = 128e^{\frac{11\pi i}{6}} = 128\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = 128\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

By geometrical considerations the side length of the equilateral triangle is

$128 \times 2\cos\frac{\pi}{6} = 128\sqrt{3}$ hence the area is $\frac{1}{2} \times 128\sqrt{3} \times 128\sqrt{3}\sin\frac{\pi}{3} = 12288\sqrt{3}$

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**6** We can represent the action of 'moving forward one and then turning to the right by $\frac{2\pi}{9}$' by the function that acts on complex numbers by $f(z) = \mathrm{e}^{-\frac{2\pi i}{9}}(z+1)$ without loss of generality we can assume the ant starts at the origin, then the final position of the ant after performing this 4 times is $f^4(0)$ which we shall now compute, we have

$f(0) = \mathrm{e}^{-\frac{2\pi i}{9}}$

So

$f^2(0) = f(\mathrm{e}^{-\frac{2\pi i}{9}}) = \mathrm{e}^{-\frac{2\pi i}{9}}(\mathrm{e}^{-\frac{2\pi i}{9}}+1) = \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}$

So

$$f^3(0) = f\left(\mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}\right) = \mathrm{e}^{-\frac{2\pi i}{9}}\left(\mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}} + 1\right) = \mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}$$

$$f^4(0) = \mathrm{e}^{-\frac{2\pi i}{9}}\left(\mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}} + 1\right) = \mathrm{e}^{-\frac{8\pi i}{9}} + \mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}}$$

Let $z = f^4(0)$ then this is just a geometric series which we can sum to give

$$z = \mathrm{e}^{-\frac{8\pi i}{9}} + \mathrm{e}^{-\frac{6\pi i}{9}} + \mathrm{e}^{-\frac{4\pi i}{9}} + \mathrm{e}^{-\frac{2\pi i}{9}} = \frac{\mathrm{e}^{-\frac{8\pi i}{9}}\left(1 - \mathrm{e}^{\frac{8\pi i}{9}}\right)}{1 - \mathrm{e}^{\frac{2\pi i}{9}}}$$

Hence the distance is the modulus of this complex number and we have

$$|z|^2 = \left|\frac{\mathrm{e}^{-\frac{8\pi i}{9}}\left(1 - \mathrm{e}^{\frac{8\pi i}{9}}\right)}{1 - \mathrm{e}^{\frac{2\pi i}{9}}}\right|^2 = \frac{\left|1 - \mathrm{e}^{\frac{8\pi i}{9}}\right|^2}{\left|1 - \mathrm{e}^{\frac{2\pi i}{9}}\right|^2} = \frac{\left(1 - \cos\frac{8\pi}{9}\right)^2 + \sin^2\frac{8\pi}{9}}{\left(1 - \cos\frac{2\pi}{9}\right)^2 + \sin^2\frac{2\pi}{9}} = \frac{2 - 2\cos\frac{8\pi}{9}}{2 - 2\cos\frac{2\pi}{9}} = \frac{\sin^2\frac{4\pi}{9}}{\sin^2\frac{\pi}{9}}$$

Hence the distance from the starting position is

$$|z| = \frac{\sin\frac{4\pi}{9}}{\sin\frac{\pi}{9}}$$

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