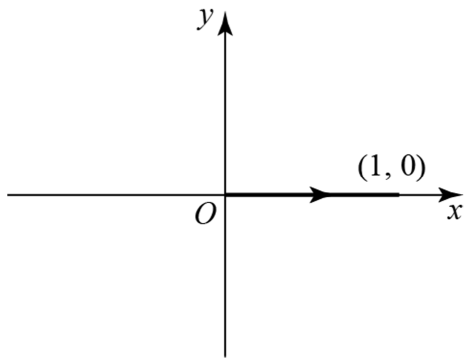
Complex numbers 1F

1 **a** $z^4 - 1 = 0$

$z^4 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^4 = 1(\cos 0 + \mathrm{i}\sin 0)$

$z^4 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + \mathrm{i}\sin(2k\pi)]^{\frac{1}{4}}$

$z = \cos\left(\frac{2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{4}\right)$

de Moivre's Theorem.

$z = \cos\left(\frac{k\pi}{2}\right) + \mathrm{i}\sin\left(\frac{k\pi}{2}\right)$

$k = 0,\ z = \cos 0 + \mathrm{i}\sin 0 = 1$

$k = 1,\ z = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2} = \mathrm{i}$

$k = 2,\ z = \cos\pi + \mathrm{i}\sin\pi = -1$

$k = -1,\ z = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = -\mathrm{i}$

Therefore, $z = 1,\ \mathrm{i},\ -1,\ -\mathrm{i}$

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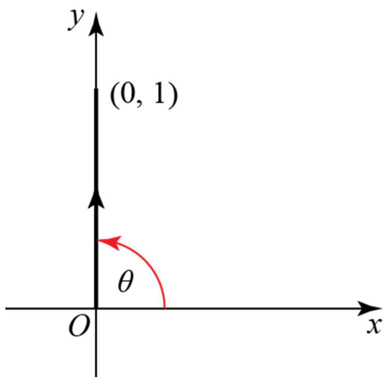
1 b $z^3 - \mathrm{i} = 0$

$z^3 = \mathrm{i}$



for i, $r = 1$ and $\theta = \frac{\pi}{2}$

So $z^3 = 1\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$

$z^3 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$

Hence, $z = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2k\pi\right)\right]^{\frac{1}{3}}$

$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)$ — de Moivre's Theorem.

$z = \cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

$\therefore k = 0,\ z = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$k = 1,\ z = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

$k = -1,\ z = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = 0 - \mathrm{i}$

Therefore, $z = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i},\ -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i},\ -\mathrm{i}$

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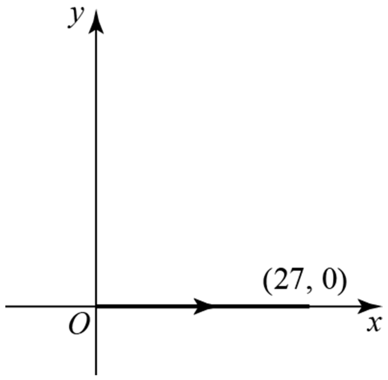
1 c $z^3 = 27$



for $27,\ r = 27$ and $\theta = 0$

So $z^3 = 27(\cos 0 + \mathrm{i}\sin 0)$

$z^3 = 27[\cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi)] \quad k \in \mathbb{Z}$

Hence, $z = [27(\cos 2k\pi) + \mathrm{i}\sin(2k\pi)]^{\frac{1}{3}}$

de Moivre's Theorem.

$$z = 3\left[\cos\left(\frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{3}\right)\right]$$

$k = 0;\ z = 3(\cos 0 + \mathrm{i}\sin 0) = 3$

$$k = 1;\ z = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i}$$

$$k = -1;\ z = 3\left(\cos\left(\frac{-2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$$

Therefore, $z = 3,\ -\frac{3}{2} + \frac{3\sqrt{3}}{2}\mathrm{i},\ -\frac{3}{2} - \frac{3\sqrt{3}}{2}\mathrm{i}$

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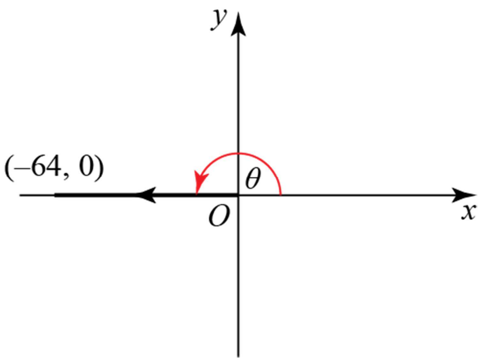
1 d $z^4 + 64 = 0$

$z^4 = -64$



for -64, $r = 64$ and $\theta = \pi$

So $z^4 = 64(\cos\pi + \mathrm{i}\sin\pi)$

$z^4 = 64(\cos\,(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [64(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 64^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$$k = 0;\ z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) = 2 + 2\mathrm{i}$$

$$k = 1;\ z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{2}\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) = -2 + 2\mathrm{i}$$

$$k = -1;\ z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) = 2 - 2\mathrm{i}$$

$$k = -2;\ z = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) = -2 - 2\mathrm{i}$$

Therefore, $\mathrm{z} = 2 + 2\mathrm{i},\ -2 + 2\mathrm{i},\ 2 - 2\mathrm{i}, -2 - 2\mathrm{i}$

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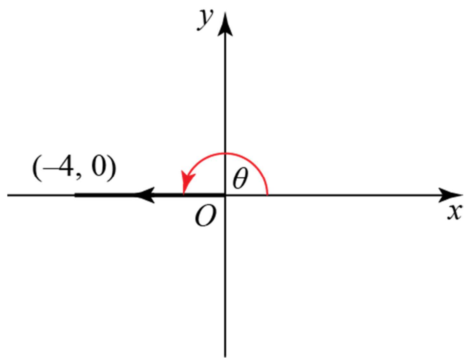
1 e $z^4 + 4 = 0$

$z^4 = -4$



for -4, $r = 4$ and $\theta = \pi$

So $z^4 = 4(\cos\pi + i\sin\pi)$

$z^4 = 4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [4(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$$k = 0;\ z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$$

$$k = 1;\ z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -1 + i$$

$$k = -1;\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 1 - i$$

$$k = -2;\ z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$$

Therefore, $z = 1 + i,\ -1 + i,\ 1 - i,\ -1 - i$

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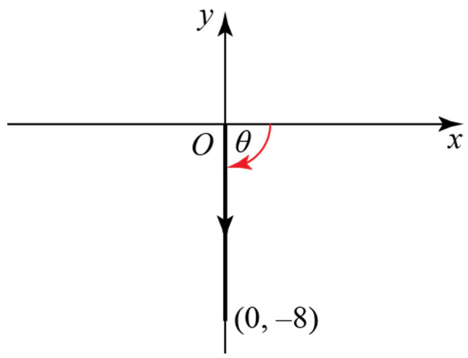
1 f $z^3 + 8\mathrm{i} = 0$

$z^3 = -8\mathrm{i}$



for $-8\mathrm{i}$, $r = 8$, $\theta = -\frac{\pi}{2}$

So $z^3 = 8\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$z^4 = 8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right)\right)$

de Moivre's Theorem.

$z = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$

$k = 0;\ z = 2\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = \sqrt{3} - \mathrm{i}$

$k = 1;\ z = 2\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 2(0 + \mathrm{i}) = 2\mathrm{i}$

$k = -1;\ z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = -\sqrt{3} - \mathrm{i}$

Therefore, $z = \sqrt{3} - \mathrm{i},\ 2\mathrm{i},\ -\sqrt{3} - \mathrm{i}$

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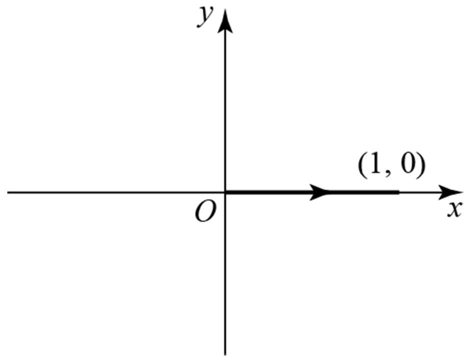
2 a $z^7 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^7 = 1(\cos 0 + \mathrm{i}\sin 0)$

$z^7 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = (\cos(2k\pi) + \mathrm{i}\sin(2k\pi))^{\frac{1}{7}}$

$z = \cos\left(\frac{2k\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{7}\right)$ 　 de Moivre's Theorem.

$k = 0,\ z = \cos 0 + \mathrm{i}\sin 0$

$k = 1,\ z = \cos\left(\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2\pi}{7}\right)$

$k = 2,\ z = \cos\left(\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(\frac{4\pi}{7}\right)$

$k = 3,\ z = \cos\left(\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(\frac{6\pi}{7}\right)$

$k = -1,\ z = \cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right)$

$k = -2,\ z = \cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$

$k = -3,\ z = \cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$

Therefore, $z = \cos 0 + \mathrm{i}\sin 0,\ \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$

$\cos\frac{4\pi}{7} + \mathrm{i}\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7} + \mathrm{i}\sin\frac{6\pi}{7}$

$\cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right),\ \cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$

$\cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$

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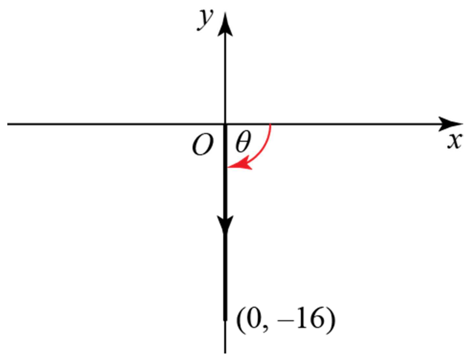
2 b $z^4 + 16\text{i} = 0$

$z^4 = -16\text{i}$



for -16i, $r = 16$ and $\theta = -\frac{\pi}{2}$

So $z^4 = 16\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$z^4 = 16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \text{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$

$z = 16^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right) + \text{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)\right)$

de Moivre's Theorem.

$z = \left(\cos\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)\right)$

$k = 0,\ z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right)$

$k = 1,\ z = 2\left(\cos\frac{3\pi}{8} + \text{i}\sin\frac{3\pi}{8}\right)$

$k = 2,\ z = 2\left(\cos\frac{7\pi}{8} + \text{i}\sin\frac{7\pi}{8}\right)$

$k = -1,\ z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

Therefore, $z = 2\left(\cos\left(-\frac{\pi}{8}\right) + \text{i}\sin\left(-\frac{\pi}{8}\right)\right), 2\left(\cos\left(\frac{3\pi}{8}\right) + \text{i}\sin\left(\frac{3\pi}{8}\right)\right)$

$2\left(\cos\left(\frac{7\pi}{8}\right) + \text{i}\sin\left(\frac{7\pi}{8}\right)\right), 2\left(\cos\left(-\frac{5\pi}{8}\right) + \text{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

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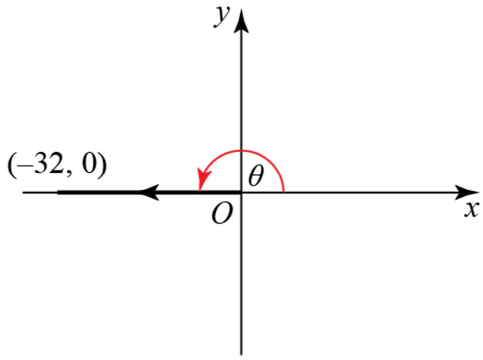
2 c $z^5 + 32 = 0$

$z^5 = -32$



for -32, $r = 32$ and $\theta = \pi$

So $z^5 = 32(\cos \pi + \mathrm{i}\sin \pi)$

$z^5 = 32(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [32(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi))]^{\frac{1}{5}}$

$$z = 32^{\frac{1}{5}}\left(\cos\left(\frac{\pi + 2k\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{5}\right)\right)$$

$$z = 2\left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)\right)$$

de Moivre's Theorem.

$k = 0$, $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$

$k = 1$, $z = 2\left(\cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}\right)$

$k = 1$, $z = 2(\cos\pi + \mathrm{i}\sin\pi)$

$k = 2$, $z = 2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$

$k = -1$, $z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

$k = -2$, $z = 2\left(\cos\left(-\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{5}\right)\right)$

Therefore, $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$, $2\left(\cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}\right)$,

$2(\cos\pi + \mathrm{i}\sin\pi)$, $2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$,

$2\left(\cos\left(-\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{5}\right)\right)$

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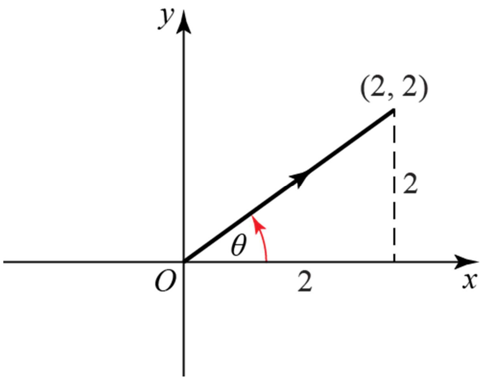
2 d $z^3 = 2 + 2\text{i}$



$r = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

So $z^3 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

$z^3 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{4} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + \text{i}\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + \text{i}\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right)$

de Moivre's Theorem.

$z = \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$

$k = 0,\ z = \sqrt{2}\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right)$

$k = -1,\ z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \text{i}\sin\left(\frac{-7\pi}{12}\right)\right)$

Therefore, $z = \sqrt{2}\left(\cos\frac{\pi}{12} + \text{i}\sin\frac{\pi}{12}\right),\ \sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right),$

$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \text{i}\sin\left(\frac{-7\pi}{12}\right)\right)$

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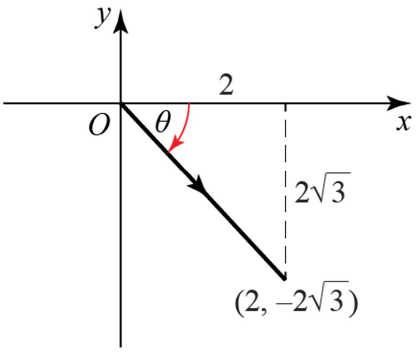
2 e $z^4 + 2\sqrt{3}\,\mathrm{i} = 2$

$z^4 = 2 - 2\sqrt{3}\,\mathrm{i}$



$$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\theta = -\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\text{So } z^4 = 4\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z^4 = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[4\left[\cos\left(-\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right]\right]^{\frac{1}{4}}$$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$$

$$k = 0,\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right)$$

$$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right)$$

$$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right)$$

$$k = -1, z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$$

$$\text{Therefore, } z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(\frac{5\pi}{12}\right)\right),$$

$$\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(\frac{11\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$$

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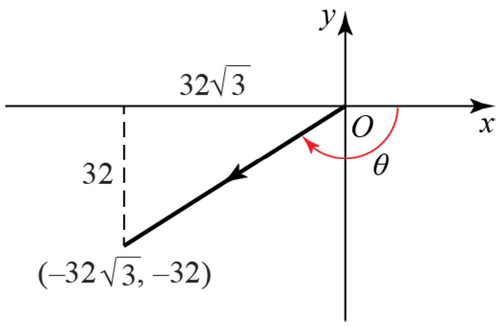
2 f $z^3 + 32\sqrt{3} + 32\text{i} = 0$

$z^3 = -32\sqrt{3} - 32\text{i}$



$r = \sqrt{(-32\sqrt{3})^2 + (-32)^2} = \sqrt{3072 + 1024} = \sqrt{4096} = 64$

$\theta = -\pi + \tan^{-1}\left(\frac{32}{32\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

So $z^4 = 64\left(\cos\left(-\frac{5\pi}{6}\right) + \text{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$z^3 = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \text{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + \text{i}\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = 64^{\frac{1}{3}}\left(\cos\left(\frac{\frac{-5\pi}{6} + 2k\pi}{3}\right) + \text{i}\sin\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right)\right)$ — de Moivre's Theorem.

$z = 4\left(\cos\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + \text{i}\sin\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right)$

$k = 0,\ z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \text{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

$k = 1,\ z = 4\left(\cos\left(\frac{7\pi}{18}\right) + \text{i}\sin\left(\frac{7\pi}{18}\right)\right)$

$k = -1,\ z = 4\left(\cos\left(-\frac{17\pi}{18}\right) + \text{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

Therefore, $z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + \text{i}\sin\left(-\frac{5\pi}{18}\right)\right),\ 4\left(\cos\left(\frac{7\pi}{18}\right) + \text{i}\sin\left(\frac{7\pi}{18}\right)\right),$

$4\left(\cos\left(-\frac{17\pi}{18}\right) + \text{i}\sin\left(-\frac{17\pi}{18}\right)\right)$

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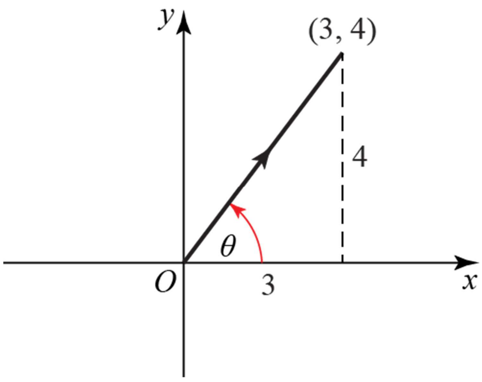
3 a $z^4 = 3 + 4\text{i}$



$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927295\ldots$

So $z^4 = 5\text{e}^{\text{i}(0.927295\ldots)}$

$z^4 = 5\text{e}^{\text{i}(0.927295\ldots+2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [5\text{e}^{\text{i}(0.927295\ldots+2k\pi)}]^{\frac{1}{4}}$

$= 5^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{0.927295\ldots+2k\pi}{4}\right)}$

de Moivre's Theorem.

$= 5^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{0.927295\ldots}{4}+\frac{k\pi}{2}\right)}$

$k = 0,\ z = 5^{\frac{1}{4}}\text{e}^{\text{i}(0.2318\ldots)}$

$k = 1,\ z = 5^{\frac{1}{4}}\text{e}^{\text{i}(1.8026\ldots)}$

$k = -1,\ z = 5^{\frac{1}{4}}\text{e}^{\text{i}(-1.3389\ldots)}$

$k = -2,\ z = 5^{\frac{1}{4}}\text{e}^{\text{i}(-2.9097\ldots)}$

Therefore, $z = 5^{\frac{1}{4}}\text{e}^{0.23\text{i}},\ 5^{\frac{1}{4}}\text{e}^{1.80\text{i}},\ 5^{\frac{1}{4}}\text{e}^{-1.34\text{i}},\ 5^{\frac{1}{4}}\text{e}^{-2.91\text{i}}$

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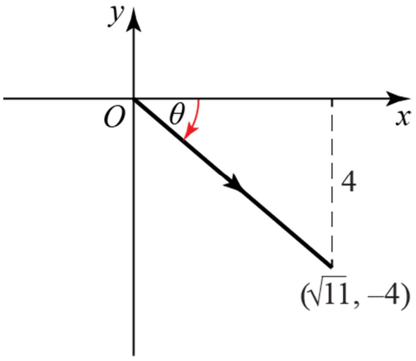
3 b $z^3 = \sqrt{11} + 4\text{i}$



$$r = \sqrt{(\sqrt{11})^2 + (-4)^2} = \sqrt{11+16} = \sqrt{27}$$

$$\theta = -\tan^{-1}\left(\frac{4}{\sqrt{11}}\right) = 0.878528\ldots$$

So, $z^3 = \sqrt{27}\,\text{e}^{\text{i}(-0.878528\ldots)}$

$z^3 = \sqrt{27}\,\text{e}^{\text{i}(-0.878528\ldots+2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [\sqrt{270}\,\text{e}^{\text{i}(-0.878528\ldots+2k\pi)}]^{\frac{1}{3}}$

de Moivre's Theorem.

$= (\sqrt{27})^{\frac{1}{3}}\text{e}^{\text{i}\left(\frac{-0.878528\ldots+2k\pi}{3}\right)}$

$= \sqrt{3}\,\text{e}^{\text{i}\frac{-0.878528\ldots}{3}+\frac{2k\pi}{3}}$

$k = 0,\ z = \sqrt{3}\,\text{e}^{\text{i}(-0.2928\ldots)}$

$k = 1,\ z = \sqrt{3}\,\text{e}^{\text{i}(1.8015\ldots)}$

$k = -1,\ z = \sqrt{3}\,\text{e}^{\text{i}(-2.3872\ldots)}$

Therefore, $z = \sqrt{3}\,\text{e}^{-0.29\text{i}},\ \sqrt{3}\,\text{e}^{1.80\text{i}},\ \sqrt{3}\,\text{e}^{-2.39\text{i}}$

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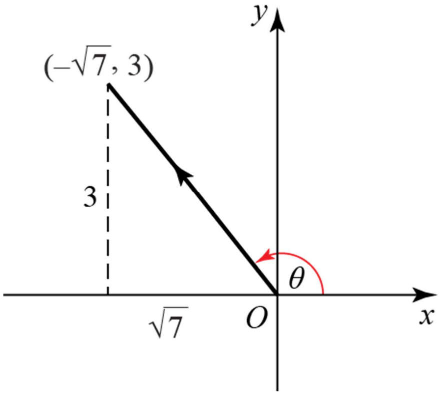
3 c $z^4 = -\sqrt{7} + 3\text{i}$



$r = \sqrt{(-\sqrt{7})^2 + 3^2} = \sqrt{7+9} = \sqrt{16} = 4$

$\theta = \pi - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = 2.293530\ldots$

So, $z^4 = 4\text{e}^{\text{i}(2.293530\ldots)}$

$z^4 = 4\text{e}^{\text{i}(2.293530\ldots + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [4\,\text{e}^{\text{i}(2.293530\ldots + 2k\pi)}]^{\frac{1}{4}}$

$= 4^{\frac{1}{4}}\,\text{e}^{\text{i}\left(\frac{2.293530\ldots + 2k\pi}{4}\right)}$ — de Moivre's Theorem.

$= \sqrt{2}\,\text{e}^{\text{i}\left(\frac{2.293530\ldots}{4} + \frac{k\pi}{2}\right)}$

$k = 0,\ z = \sqrt{2}\,\text{e}^{\text{i}(0.5733\ldots)}$

$k = 1,\ z = \sqrt{2}\,\text{e}^{\text{i}(2.1441\ldots)}$

$k = -1,\ z = \sqrt{2}\,\text{e}^{\text{i}(-0.9974\ldots)}$

$k = -2,\ z = \sqrt{2}\,\text{e}^{\text{i}(-2.5682\ldots)}$

Therefore, $z = \sqrt{2}\,\text{e}^{0.57\text{i}},\ z = \sqrt{2}\,\text{e}^{2.14\text{i}},\ z = \sqrt{2}\,\text{e}^{-1.00\text{i}},\ z = \sqrt{2}\,\text{e}^{-2.57\text{i}}$

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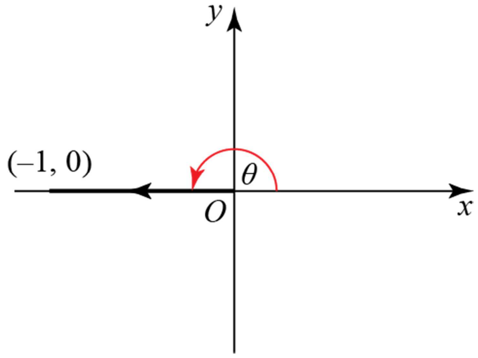
4 a $(z+1)^3 = -1$



For -1, $r = 1$ and $\theta = \pi$

So, $(z+1)^3 = 1(\cos\pi + \mathrm{i}\sin\pi)$

$(z+1)^3 = \cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z+1 = [\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi)]^{\frac{1}{3}}$

de Moivre's Theorem.

$$z+1 = \cos\left(\frac{\pi + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{3}\right)$$

$$z+1 = \cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)$$

$$k = 0,\ z+1 = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

$$\Rightarrow z = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$$

$$k = 1,\ z+1 = \cos\pi + \mathrm{i}\sin\pi = -1 + 0\mathrm{i}$$

$$\Rightarrow z = -2$$

$$k = -1,\ z+1 = \cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

$$\Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$$

Therefore, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ -2,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

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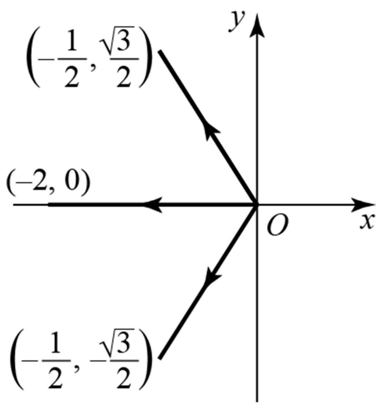
4 b

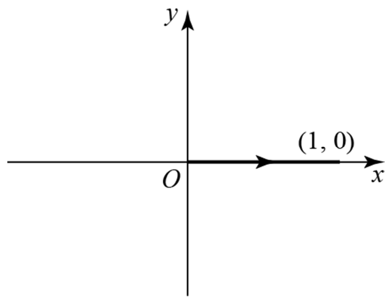


c The solution to $w^3 = -1$, lie on a circle centre (0, 0), radius 1.
As $w = z + 1$, then the three solutions for z are the three solutions for w translated by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Hence the three points (the solutions for z), lie on a circle centre (–1, 0), radius 1.

5 a $z^5 - 1 = 0$

$z^5 = 1$



For 1, $r = 1$ and $\theta = 0$

So, $z^5 = 1(\cos 0 + i\sin 0)$

$z^5 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{5}}$

$z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$

de Moivre's Theorem.

$k = 0,\ z_1 = \cos 0 + i\sin 0 = 1 + i(0) = 1$

$k = 1,\ z_2 = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$

$k = 2,\ z_3 = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$

$k = -1,\ z_4 = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$k = -2,\ z_5 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

Therefore $z = 1,\ \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right),\ \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right),$

$\cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right),\ \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

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5 b So, $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$$1+\cos\left(\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)+\mathrm{i}\sin\left(\frac{4\pi}{5}\right)$$
$$+\cos\left(-\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{5}\right)+\cos\left(-\frac{4\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{4\pi}{5}\right)=0$$

$$\Rightarrow 1+\cos\left(\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)+\mathrm{i}\sin\left(\frac{4\pi}{5}\right)$$
$$+\cos\left(\frac{2\pi}{5}\right)-\mathrm{i}\sin\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)-\mathrm{i}\sin\left(\frac{4\pi}{5}\right)=0$$

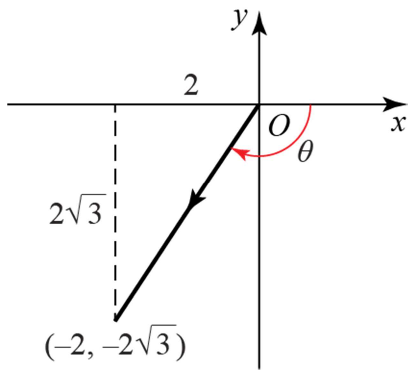
$$1+2\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{4\pi}{5}\right)=0$$

$$2\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{4\pi}{5}\right)=-1$$

$$2\left(\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)\right)=-1$$

$$\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)=-\frac{1}{2}\quad\text{(as required)}$$

6 a $-2-2\sqrt{3}\,\mathrm{i}$.



modulus $= r\sqrt{(-2)^2+(-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

argument $= \theta = 2\pi + \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

Therefore, $r = 4$, $\theta = -\frac{2\pi}{3}$

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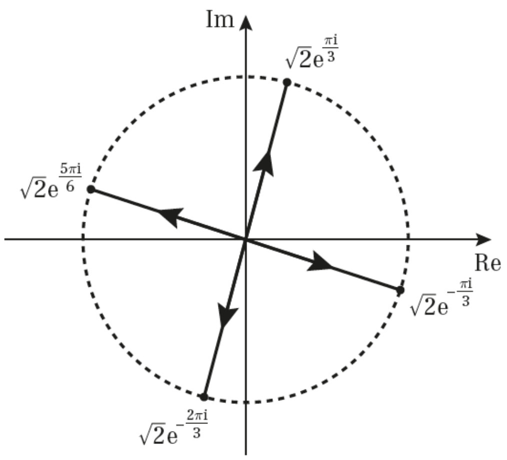
6 b We have $w = -2 - 2i\sqrt{3} = 4e^{-\frac{2\pi i}{3}}$ we wish to solve

$z^4 + 2 + 2i\sqrt{3} = 0$ which is equivalent to solving $z^4 = -2 - 2i\sqrt{3} = 4e^{-\frac{2\pi i}{3}}$

The solution has the form $z = \sqrt[4]{4}e^{i\theta}$ where the argument satisfies

$4\theta = -\frac{2\pi}{3} + 2k\pi$ for $k \in \mathbb{Z}$ hence the 4 distinct solutions have arguments given by

$\theta = -\frac{\pi}{6} + \frac{k\pi}{2}$ for $k = -1, 0, 1, 2$, so $\theta = \sqrt{2}e^{-\frac{2\pi i}{3}}, \sqrt{2}e^{-\frac{\pi i}{6}}, \sqrt{2}e^{\frac{\pi i}{3}}, \sqrt{2}e^{\frac{5\pi i}{6}}$



7 We wish to solve $z^4 = 2(1 - i\sqrt{3})$, to start off let $w = 2(1 - i\sqrt{3})$ then if we write $w = re^{i\theta}$ then

$r = \sqrt{4 + 12} = 4$

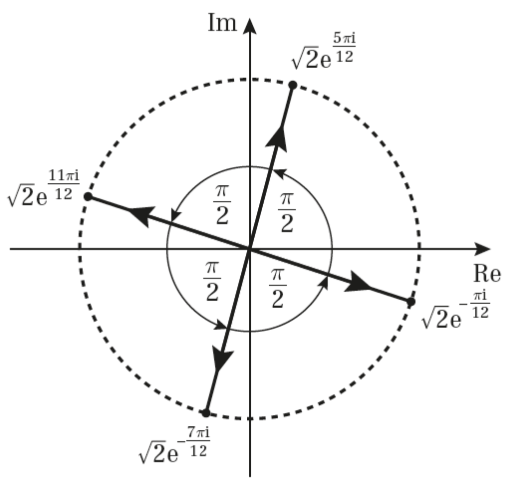
$\tan\theta = -\sqrt{3}$

So $\theta = -\frac{\pi}{3}$ and hence $w = 4e^{-\frac{\pi i}{3}}$ now going back to the original equation we have

$z^4 = 4e^{-\frac{\pi i}{3}}$ so that $z = \sqrt[4]{4}e^{i\theta}$ where the argument must satisfy $4\theta = -\frac{\pi}{3} + 2k\pi$ for $k \in \mathbb{Z}$

So $\theta = -\frac{\pi}{12} + \frac{k\pi}{2}$ hence the 4 distinct roots are given by these values of θ for $k = 0, 1, 2, 3$

Hence $\theta = \sqrt{2}e^{-\frac{7\pi i}{12}}, \sqrt{2}e^{-\frac{\pi i}{12}}, \sqrt{2}e^{\frac{5\pi i}{12}}, \sqrt{2}e^{\frac{11\pi i}{12}}$



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8 **a** Let $z=\sqrt{6}+i\sqrt{2}$ then if we write $z=re^{i\theta}$ then we have $r=\sqrt{6+2}=2\sqrt{2}$ and

$\tan\theta=\frac{\sqrt{2}}{\sqrt{6}}=\frac{1}{\sqrt{3}}$ so that $\theta=\frac{\pi}{6}$ so we have $z=2\sqrt{2}e^{\frac{\pi i}{6}}$

b We wish to solve

$$\omega^3=z^4=\left(2\sqrt{2}e^{\frac{\pi i}{6}}\right)^4=64e^{\frac{2\pi i}{3}}$$

If we write $\omega=re^{i\theta}$ then $r=\sqrt[3]{64}=4$ and $3\theta=\frac{2\pi}{3}+2k\pi$ for $k\in\mathbb{Z}$ so that the distinct solutions correspond to $\theta=\frac{2\pi}{9}+\frac{2k\pi}{3}$ for $k=0,1,2$, explicitly these are

$$\omega=4e^{\frac{2\pi i}{9}}=4\cos\frac{2\pi}{9}+4i\sin\frac{2\pi}{9}$$

$$\omega=4e^{\frac{8\pi i}{9}}=4\cos\frac{8\pi}{9}+4i\sin\frac{8\pi}{9}$$

$$\omega=4e^{\frac{14\pi i}{9}}=4e^{\frac{-4\pi i}{9}}=4\cos\frac{-4\pi}{9}+4i\sin\frac{-4\pi}{9}=4\cos\frac{4\pi}{9}-4i\sin\frac{4\pi}{9}$$

9 **a** We wish to solve $1+z+z^2+z^3+z^4+z^5+z^6+z^7=0$ note that the left hand side is a geometric series so if we apply the formula for its sum we get

$$\frac{1-z^8}{1-z}=0$$

Hence the equation is equivalent to $z^8=1$ whose solutions are the 8th roots of unity

$z=e^{\frac{2k\pi i}{8}}$ for $0\le k\le 7$ so $z=e^{\frac{\pi i}{4}},i,e^{\frac{3\pi i}{4}},-1,e^{\frac{5\pi i}{4}},-i,e^{\frac{7\pi i}{4}}$

b Now if we have $z^2+1=0$ then $z^2=-1$ so that $z^8=(-1)^4=1$ and hence z is a solution to z^8-1 hence z^2+1 is a factor of z^8-1 but we also have

$$z^8-1=(1-z)(1+z+z^2+z^3+z^4+z^5+z^6+z^7)$$

And since $(1-z)$ is not a factor of z^2+1 it must be the case that z^2+1 is a factor of $1+z+z^2+z^3+z^4+z^5+z^6+z^7$

In the second case we have $z^4+1=0$ and so $z^8=(-1)^2=1$ hence we have that z^4+1 is a factor of z^8-1 and since $(1-z)$ is not a factor of z^4+1 the same argument as above means z^4+1 must be a factor of z^8-1

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Challenge

a We wish to solve $z^6 = 1$, writing $z = e^{i\theta}$ implies we must have $\theta = \frac{2k\pi}{6}$ for $k \in \mathbb{Z}$ hence the 6 distinct solutions in the range $-\pi < \theta \le \pi$ are given by

$$z = e^{-\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}}, 1, e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, -1$$

b We wish to solve $(z+1)^6 = z^6$ rearranging gives

$$\frac{z^6}{(z+1)^6} = 1$$

$$\left(\frac{z}{z+1}\right)^6 = 1$$

Hence $\frac{z}{z+1} = e^{\frac{k\pi i}{3}}$ for some $1 \le k \le 6$ further rearranging gives

$$z = (z+1)e^{\frac{k\pi i}{3}}$$

$$\left(1 - e^{\frac{k\pi i}{3}}\right)z = e^{\frac{k\pi i}{3}}$$

So

$$z = \frac{e^{\frac{k\pi i}{3}}}{1 - e^{\frac{k\pi i}{3}}} = \frac{\left(1 - e^{-\frac{k\pi i}{3}}\right)e^{\frac{k\pi i}{3}}}{\left(1 - e^{\frac{k\pi i}{3}}\right)\left(1 - e^{-\frac{k\pi i}{3}}\right)} = \frac{\left(1 - e^{-\frac{k\pi i}{3}}\right)e^{\frac{k\pi i}{3}}}{\left(1 - \cos\frac{k\pi}{3}\right)^2 + \sin^2\frac{k\pi}{3}} = \frac{e^{\frac{k\pi i}{3}} - 1}{2 - 2\cos\frac{k\pi}{3}}$$

$$= \frac{\cos\frac{k\pi}{3} - 1 + i\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}} = -\frac{1}{2} + i\frac{\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}}$$

Now by the double angle formula we have

$$1 - \cos\frac{k\pi}{3} = 1 - \cos^2\frac{k\pi}{6} + \sin^2\frac{k\pi}{6} = 2\sin^2\frac{k\pi}{6}$$

$$\sin\frac{k\pi}{3} = 2\cos\frac{k\pi}{6}\sin\frac{k\pi}{6}$$

Hence

$$z = -\frac{1}{2} + i\frac{\sin\frac{k\pi}{3}}{2 - 2\cos\frac{k\pi}{3}} = -\frac{1}{2} + i\frac{2\cos\frac{k\pi}{6}\sin\frac{k\pi}{6}}{42\sin^2\frac{k\pi}{6}} = -\frac{1}{2} + \frac{i}{2}\cot\frac{k\pi}{6}$$

and take $k = 1, 2, 3, 4, 5$

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