## Complex numbers 1E

**1 a** $z = e^{\frac{\pi i}{n}}$ we have

$$1+z+z^2+...+z^{2n-1} = \frac{1-z^{2n}}{1-z} = \frac{1-e^{2\pi i}}{1-e^{\frac{\pi i}{n}}} = \frac{1-1}{1-e^{\frac{\pi i}{n}}} = 0$$

**b**

$$1+z+z^2+...+z^n = \frac{1-z^{n+1}}{1-z} = \frac{1-e^{\frac{(n+1)\pi i}{n}}}{1-e^{\frac{\pi i}{n}}} = \frac{1+e^{\frac{\pi i}{n}}}{e^{\frac{\pi i}{2n}}\left(e^{-\frac{\pi i}{2n}} - e^{\frac{\pi i}{2n}}\right)} = \frac{e^{\frac{\pi i}{2n}}\left(e^{-\frac{\pi i}{2n}} + e^{\frac{\pi i}{2n}}\right)}{e^{\frac{\pi i}{2n}}\left(e^{-\frac{\pi i}{2n}} - e^{\frac{\pi i}{2n}}\right)}$$

$$= \frac{e^{-\frac{\pi i}{2n}} + e^{\frac{\pi i}{2n}}}{e^{-\frac{\pi i}{2n}} - e^{\frac{\pi i}{2n}}} = \frac{2\cos\frac{\pi}{2n}}{-2i\sin\frac{\pi}{2n}} = i\cot\frac{\pi}{2n}$$

**2** $z = e^{\frac{\pi i}{2}}$ then we have

$$\sum_{r=0}^{12} z^r = \frac{1-z^{13}}{1-z} = \frac{1-z}{1-z} = 1$$

since $z^4 = 1$

**3** Let $z = (1+i) = \sqrt{2}e^{\frac{\pi i}{4}}$ then

$$\sum_{r=0}^{7}(1+i)^r = \sum_{r=0}^{7}\left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^r = \frac{1-\left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^8}{1-\sqrt{2}e^{\frac{\pi i}{4}}} = \frac{1-16}{1-\sqrt{2}e^{\frac{\pi i}{4}}} = \frac{-15}{1-(1+i)} = \frac{15}{i} = -15i$$

**4 a** We have

$$C = 1+\frac{1}{3}\cos\theta+\frac{1}{9}\cos 2\theta+\frac{1}{27}\cos 3\theta+...$$

$$S = \frac{1}{3}\sin\theta+\frac{1}{9}\sin 2\theta+\frac{1}{27}\sin 3\theta$$

So that

$$C+iS = 1+\frac{1}{3}(\cos\theta+i\sin\theta)+\frac{1}{9}(\cos 2\theta+i\sin 2\theta)+\frac{1}{27}(\cos 2\theta+i\sin 2\theta)+...$$

$$= 1+\frac{1}{3}e^{i\theta}+\frac{1}{9}e^{2i\theta}+\frac{1}{27}e^{3i\theta}+...$$

$$= \sum_{r=0}^{\infty}\frac{1}{3^r}e^{ir\theta} = \frac{1}{1-\frac{1}{3}e^{i\theta}} = \frac{3}{3-e^{i\theta}}$$

**4 b** We have

$$\frac{3}{3-e^{i\theta}}=\frac{3\left(3-e^{-i\theta}\right)}{\left(3-e^{i\theta}\right)\left(3-e^{-i\theta}\right)}=\frac{9-3e^{-i\theta}}{10-3\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{9-3(\cos\theta-i\sin\theta)}{10-3(2\cos\theta)}$$

$$=\frac{9-3\cos\theta+3i\sin\theta)}{10-3(2\cos\theta)}$$

Equating real and imaginary parts then gives

$$C=\frac{9-3\cos\theta}{10-6\cos\theta}$$

$$S=\frac{3\sin\theta}{10-6\cos\theta}$$

**5 a** We have

$$P=1+\cos\theta+\cos 2\theta+\cos 3\theta+\ldots+\cos 11\theta+\cos 12\theta$$

$$Q=\sin\theta+\sin 2\theta+\sin 3\theta+\ldots+\sin 11\theta+\sin 12\theta$$

So that

$$P+iQ=1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+\ldots+e^{11i\theta}+e^{12i\theta}$$

$$=\frac{1-e^{13i\theta}}{1-e^{i\theta}}=\frac{e^{\frac{13i\theta}{2}}\left(e^{-\frac{13i\theta}{2}}-e^{\frac{13i\theta}{2}}\right)}{e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}\right)}=\frac{e^{6i\theta}\left(e^{-\frac{13i\theta}{2}}-e^{\frac{13i\theta}{2}}\right)}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}$$

**b** We have

$$P+iQ=\frac{e^{6i\theta}\left(e^{-\frac{13i\theta}{2}}-e^{\frac{13i\theta}{2}}\right)}{e^{\frac{i\theta}{2}}-e^{-\frac{i\theta}{2}}}=\frac{e^{6i\theta}\times 2i\sin\frac{13\theta}{2}}{2i\sin\frac{\theta}{2}}=\sin\frac{13\theta}{2}\operatorname{cosec}\frac{\theta}{2}\left(\cos 6\theta+i\sin 6\theta\right)$$

Hence equating real and imaginary parts gives

$$P=\sin\frac{13\theta}{2}\operatorname{cosec}\frac{\theta}{2}\cos 6\theta$$

$$Q=\sin\frac{13\theta}{2}\operatorname{cosec}\frac{\theta}{2}\sin 6\theta$$

**c** $P+iQ$ is real when $Q=0$ i.e. when $\sin 6\theta\operatorname{cosec}\frac{\theta}{2}\sin\frac{13\theta}{2}=0$

$\operatorname{cosec}\frac{\theta}{2}$ can never be zero so we either have $\sin 6\theta=0$ in which case $\theta=\frac{k\pi}{6}$ for $k=1,2,3,4,5$

or we have $\sin\frac{13\theta}{2}=0$ in which case $\theta=\frac{2k\pi}{13}$ for $k=1,2,3,4,5,6$

**6 a** We have

$$C=1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\binom{n}{3}\cos 3\theta+\ldots+\binom{n}{n}\cos n\theta$$

$$S=\binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\binom{n}{3}\sin 3\theta+\ldots+\binom{n}{n}\sin n\theta$$

**6 a** So that

$$C+iS=1+\binom{n}{1}e^{i\theta}+\binom{n}{2}e^{2i\theta}+\binom{n}{3}e^{3i\theta}+\ldots+\binom{n}{n}e^{ni\theta}$$

$$=\left(1+e^{i\theta}\right)^n=\left(e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}\right)\right)^n=e^{\frac{in\theta}{2}}\left(e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}\right)^n=e^{\frac{in\theta}{2}}\left(2\cos\frac{\theta}{2}\right)^n$$

Equating real and imaginary parts then gives

$$C=\cos\frac{n\theta}{2}\left(2\cos\frac{\theta}{2}\right)^n$$

$$S=\sin\frac{n\theta}{2}\left(2\cos\frac{\theta}{2}\right)^n$$

**b** We have

$$\frac{S}{C}=\frac{\sin\frac{n\theta}{2}}{\cos\frac{n\theta}{2}}=\tan\frac{n\theta}{2}$$

**7 a** We have

$$\left(2+e^{i\theta}\right)\left(2+e^{-i\theta}\right)=4+2e^{-i\theta}+2e^{i\theta}+1=5+4\cos\theta$$

**b** We have

$$C=1-\frac{1}{2}\cos\theta+\frac{1}{4}\cos2\theta-\frac{1}{8}\cos3\theta+\ldots$$

$$S=\frac{1}{2}\sin\theta-\frac{1}{4}\sin2\theta+\frac{1}{8}\sin3\theta+\ldots$$

So that

$$C-iS=1-\frac{1}{2}e^{i\theta}+\frac{1}{4}e^{2i\theta}-\frac{1}{8}e^{3i\theta}+\ldots$$

$$=\frac{1}{1-\left(-\frac{1}{2}e^{i\theta}\right)}=\frac{1}{1+\frac{1}{2}e^{i\theta}}=\frac{2}{2+e^{i\theta}}=\frac{2\left(2+e^{-i\theta}\right)}{\left(2+e^{i\theta}\right)\left(2+e^{-i\theta}\right)}=\frac{4+2e^{-i\theta}}{5+4\cos\theta}$$

Equating real and imaginary parts then gives

$$C=\frac{4+2\cos\theta}{5+4\cos\theta}$$

$$S=\frac{2\sin\theta}{5+4\cos\theta}$$