Complex numbers 1D

1 a $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + \mathrm{i}\sin^3\theta$

de Moivre's Theorem.

$= \cos^3\theta + {}^3\mathrm{C}_1\cos^2\theta(\mathrm{i}\sin\theta) + {}^3\mathrm{C}_2\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$

Binomial expansion.

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta + 3\mathrm{i}^2\cos\theta\sin^2\theta + \mathrm{i}^3\sin^3\theta$

$= \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

Hence,
$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$

Equating the imaginary parts gives,

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Hence, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (as required)

b $(\cos\theta + \mathrm{i}\sin\theta)^5 = \cos 5\theta + \mathrm{i}\sin 5\theta$

de Moivre's Theorem.

$= \cos^5\theta + {}^5\mathrm{C}_1\cos^4\theta(\mathrm{i}\sin\theta) + {}^5\mathrm{C}_2\cos^3\theta(\mathrm{i}\sin\theta)^2 + {}^5\mathrm{C}_3\cos^2\theta(\mathrm{i}\sin\theta)^3 + {}^5\mathrm{C}_4\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$

Binomial expansion.

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta + 10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$

Hence,
$\cos 5\theta + \mathrm{i}\sin 5\theta = \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$

Equating the imaginary parts gives,

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(\cos^2\theta)^2\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta(1-\sin^2\theta) + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Hence, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (as required)

1 c $(\cos\theta + \mathrm{i}\sin\theta)^7 = \cos 7\theta + \mathrm{i}\sin 7\theta$

$$= \cos^7\theta + {}^7\mathrm{C}_1\cos^6\theta(\mathrm{i}\sin\theta) + {}^7\mathrm{C}_2\cos^5\theta(\mathrm{i}\sin\theta)^2$$
$$+ {}^7\mathrm{C}_3\cos^4\theta(\mathrm{i}\sin\theta)^3 + {}^7\mathrm{C}_4\cos^3\theta(\mathrm{i}\sin\theta)^4 + {}^7\mathrm{C}_5\cos^2\theta(\mathrm{i}\sin\theta)^5$$
$$+ {}^7\mathrm{C}_6\cos\theta(\mathrm{i}\sin\theta)^6 + (\mathrm{i}\sin\theta)^7$$
$$= \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta + 21\mathrm{i}^2\cos^5\theta\sin^2\theta$$
$$+ 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta$$
$$+ 7\mathrm{i}^6\cos\theta\sin^6\theta + \mathrm{i}^7\sin^7\theta$$

de Moivre's Theorem.

Binomial expansion.

Hence,

$$\cos 7\theta + \mathrm{i}\sin 7\theta = \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta$$
$$- 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta$$
$$- 7\cos\theta\sin^6\theta - \mathrm{i}\sin^7\theta$$

Equating the imaginary parts gives,

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$
$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2$$
$$- 7\cos\theta(1-\cos^2\theta)^3$$
$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)$$
$$- 7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$$
$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta$$
$$- 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$
$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

Hence, $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos^5\theta$ (as required)

1 d Let $z = \cos\theta + \mathrm{i}\sin\theta$

$$z + \frac{1}{z} = 2\cos\theta$$

$$\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$$

$$= z^4 + {}^4\mathrm{C}_1 z^3\left(\frac{1}{z}\right) + {}^4\mathrm{C}_2 z^2\left(\frac{1}{z}\right)^2 + {}^4\mathrm{C}_3 z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z^2\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$$

$$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

So, $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$$16\cos^4\theta = 2(\cos 4\theta + 4\cos 2\theta + 3)$$

$$\cos^4\theta = \frac{2}{16}(\cos 4\theta + 4\cos 2\theta + 3)$$

Therefore, $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (as required)

e Let $z = \cos\theta + \mathrm{i}\sin\theta$

$$z - \frac{1}{z} = 2\mathrm{i}\sin\theta$$

$$\left(z - \frac{1}{z}\right)^5 = (2\mathrm{i}\sin\theta)^5 = 32\mathrm{i}^5\sin^5\theta = 32\mathrm{i}\sin^5\theta$$

$$= z^5 + {}^5\mathrm{C}_1 z^4\left(-\frac{1}{z}\right) + {}^5\mathrm{C}_2 z^3\left(-\frac{1}{z}\right)^2 + {}^5\mathrm{C}_3 z^2\left(-\frac{1}{z}\right)^3 + {}^5\mathrm{C}_4 z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5}$$

$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$z^n + \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

$$= 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$$

So, $32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$ $(\div 2\mathrm{i})$

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Therefore, $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

2 a Let $z = \cos\theta + i\sin\theta$ then we have $z^5 + z^{-5} = 2\cos 5\theta$ hence

$\cos 5\theta = \frac{1}{2}\left(z^5 + z^{-5}\right)$

We have

$z^5 = \left(\cos\theta + i\sin\theta\right)^5$

$= \left(\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\right)$

and

$z^{-5} = \left(\cos\theta - i\sin\theta\right)^5$

$= \left(\cos^5\theta - 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta + 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta - i\sin^5\theta\right)$

Hence

$$\begin{aligned}\cos 5\theta &= \frac{1}{2}\left(2\cos^5\theta - 20\cos^3\theta\sin^2\theta + 10\cos\theta\sin^4\theta\right)\\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\\ &= \cos^5\theta - 10\cos^3\theta\left(1-\cos^2\theta\right) + 5\cos\theta(1-\cos^2\theta)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

b We have that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and we wish to solve

$\cos 5\theta + 5\cos 3\theta = 0$

Using the above identities the equation becomes

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta + 5\left(4\cos^3\theta - 3\cos\theta\right) = 0$

Which simplifies to

$16\cos^5\theta - 10\cos\theta = 0$

Hence we either have $\cos\theta = 0$ in which case $\theta = \frac{\pi}{2} = 1.571$ or we have

$16\cos^4\theta = 10$

Hence

$\cos\theta = \pm\sqrt[4]{\frac{5}{8}}$

Which implies we have $\theta = 0.475$ or $\theta = 2.666$

3 Let $z = \cos\theta + \mathrm{i}\sin\theta$

a $\left(z+\dfrac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

$z - \dfrac{1}{z} = 2\cos\theta$

$$= z^6 + {}^6\mathrm{C}_1 z^5\left(\frac{1}{z}\right) + {}^6\mathrm{C}_2 z^4\left(\frac{1}{z}\right)^2 + {}^6\mathrm{C}_3 z^3\left(\frac{1}{z}\right)^3 + {}^6\mathrm{C}_4 z^2\left(\frac{1}{z}\right)^4 + {}^6\mathrm{C}_5\, z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$= z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$$

$$= z^6 - 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\sin 2\theta) + 20$$

$z^n + \dfrac{1}{z^n} = 2\cos\theta$

So, $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ (as required)

b $\displaystyle\int_0^{\frac{\pi}{6}} \cos^6\theta\,\mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{6}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\,\mathrm{d}\theta$

$$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} + \frac{6\sin 4\theta}{4} + \frac{15\sin 2\theta}{2} + 10\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}\left[\left(\frac{\sin(\pi)}{6} + \frac{6\sin\left(\frac{2\pi}{3}\right)}{4} + \frac{15\sin\left(\frac{\pi}{3}\right)}{2} + \frac{10\pi}{6}\right) - (0)\right]$$

$$= \frac{1}{32}\left[0 + \frac{3}{2}\frac{\sqrt{3}}{2} + \frac{15}{2}\frac{\sqrt{3}}{2} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{3}{4}\sqrt{3} + \frac{15}{4}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{9}{2}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$$

$$\therefore \int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$$

$a = \dfrac{5}{96},\ b = \dfrac{9}{64}$

4 a We wish to show that

$32\cos^2\theta\sin^4\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

Let us start with the right hand side of the equation, letting $z = \cos\theta + i\sin\theta$ we have

$$\cos 6\theta = \frac{1}{2}\left(z^6 + z^{-6}\right)$$
$$= \frac{1}{2}\left(2\cos^6\theta - 30\cos^4\theta\sin^2\theta + 30\cos^2\theta\sin^4\theta - 2\sin^6\theta\right)$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$\cos 4\theta = \frac{1}{2}\left(z^4 + z^{-4}\right)$$
$$= \frac{1}{2}\left(2\cos^4\theta - 12\cos^2\theta\sin^2\theta + 2\sin^4\theta\right)$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Hence the right hand side becomes

$$\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta - 2\left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right)$$
$$-\left(\cos^2\theta - \sin^2\theta\right) + 2$$
$$= \cos^2\theta\left(1 - \sin^2\theta\right)^2 - 15\left(1 - \sin^2\theta\right)\cos^2\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \left(1 - \cos^2\theta\right)\sin^4\theta$$
$$-2\cos^4\theta + 12\cos^2\theta\sin^2\theta - 2\sin^4\theta - \cos^2\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta + \cos^2\theta - 2\cos^2\theta\sin^2\theta - 15\cos^2\theta\sin^2\theta - \sin^4\theta - 2\cos^4\theta + 12\cos^2\theta\sin^2\theta$$
$$-2\sin^4\theta - \cos^2\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - 5\cos^2\theta\sin^2\theta - 3\sin^4\theta - 2\cos^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - \cos^2\theta\left(5\sin^2\theta + 2\cos^2\theta\right) - 3\sin^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta - \left(1 - \sin^2\theta\right)\left(3\sin^2\theta + 2\right) - 3\sin^4\theta + \sin^2\theta + 2$$
$$= 32\cos^2\theta\sin^4\theta$$

b We have

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta \,\mathrm{d}\theta = \frac{1}{32}\int_0^{\frac{\pi}{3}} \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2\,\mathrm{d}\theta$$
$$= \frac{1}{32}\left[\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta - \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{32}\left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{2\pi}{3}\right) = \frac{\pi}{48}$$

5 a We wish to compute $\int_0^{\frac{\pi}{2}} \sin^6 \theta \, d\theta$, let $z = \cos\theta + i\sin\theta$ then we have

$\sin\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)$ so that

$$\sin^6\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^6 = \frac{-1}{64}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right)$$

$$= \frac{-1}{64}\left(2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\right)$$

$$= \frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

Hence the integral becomes

$$\frac{1}{32}\int_0^{\frac{\pi}{2}}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right) d\theta = \frac{1}{32}\left[\frac{-1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta - \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5\pi}{32}$$

b We wish to compute $\int_0^{\frac{\pi}{4}} \sin^2\theta \cos^4\theta \, d\theta$ we have

$$\sin^2\theta\cos^4\theta = \sin^2\theta\left(1 - \sin^2\theta\right)^2 = \sin^6\theta - 2\sin^4\theta + \sin^2\theta$$

From the previous part we know that

$$\sin^6\theta = \frac{1}{32}\left(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10\right)$$

and

$$\sin^4\theta = \left(\frac{1}{2i}\left(z - z^{-1}\right)\right)^4 = \frac{1}{16}\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$$

$$= \frac{1}{16}\left(2\cos 4\theta - 8\cos 2\theta + 6\right) = \frac{1}{32}\left(4\cos 4\theta - 16\cos 2\theta + 12\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right) = \frac{1}{32}\left(16 - 16\cos 2\theta\right)$$

Hence we have

$$\sin^2\theta\cos^4\theta = \sin^6\theta - 2\sin^4\theta + \sin^2\theta$$

$$= \frac{1}{32}\left(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2\right)$$

So the integral becomes

$$\frac{1}{32}\int_0^{\frac{\pi}{4}}\left(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2\right) d\theta = \frac{1}{32}\left[-\frac{1}{6}\sin 6\theta - \frac{1}{2}\sin 4\theta + \frac{1}{2}\sin 2\theta + 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{32}\left(\frac{1}{6} + \frac{1}{2} + \frac{\pi}{2}\right) = \frac{1}{64}\left(\frac{4}{3} + \pi\right) = \frac{4 + 3\pi}{192} = \frac{\pi}{64} + \frac{1}{48}$$

5 c We wish to compute

$$\int_0^{\frac{\pi}{6}} \sin^3\theta\cos^5\theta\,d\theta$$

Let $z=\cos\theta+i\sin\theta$ then we have

$$\sin^3\theta\cos^5\theta=\left(\frac{1}{2i}\left(z-z^{-1}\right)\right)^3\left(\frac{1}{2}\left(z+z^{-1}\right)\right)^5$$
$$=\frac{i}{256}\left(z^3-3z+3z^{-1}-z^{-3}\right)\left(z^5-5z^3+10z-10z^{-1}+5z^{-3}-z^{-5}\right)$$
$$=\frac{i}{256}\left(z^8+2z^6-2z^4-6z^2+6z^{-2}+2z^{-4}-2z^{-6}-z^{-8}\right)$$
$$=\frac{i}{256}\left(2i\left(\sin 8\theta+2\sin 6\theta-2\sin 4\theta-6\sin 2\theta\right)\right)$$
$$=\frac{-1}{128}\left(\sin 8\theta+2\sin 6\theta-2\sin 4\theta-6\sin 2\theta\right)$$

Hence the integral becomes

$$\frac{-1}{128}\int_0^{\frac{\pi}{6}}\left(\sin 8\theta+2\sin 6\theta-2\sin 4\theta-6\sin 2\theta\right)d\theta$$
$$=\frac{-1}{128}\left[\frac{-1}{8}\cos 8\theta-\frac{1}{3}\cos 6\theta+\frac{1}{2}\cos 4\theta+3\cos 2\theta\right]_0^{\frac{\pi}{6}}$$
$$=\frac{-1}{128}\left(\left(\frac{1}{16}+\frac{1}{3}-\frac{1}{4}+\frac{3}{2}\right)-\left(\frac{-1}{8}-\frac{1}{3}+\frac{1}{2}+3\right)\right)=\frac{-1}{128}\left(\frac{79}{48}-\frac{146}{48}\right)=\frac{67}{6144}$$

6 a Let $z=\cos\theta+i\sin\theta$ then we have

$$\cos 6\theta=\frac{1}{2}\left(z^6+z^{-6}\right)=\frac{1}{2}\left(\left(\cos\theta+i\sin\theta\right)^6+\left(\cos\theta-i\sin\theta\right)^6\right)$$

Noting that odd powers will cancel this simplifies to

$$=\frac{1}{2}\left(2\cos^6\theta-30\cos^4\theta\sin^2\theta+30\cos^2\theta\sin^4\theta-2\sin^6\theta\right)$$
$$=\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta$$
$$=\cos^6\theta-15\cos^4\theta\left(1-\cos^2\theta\right)+15\cos^2\theta\left(1-\cos^2\theta\right)^2-\left(1-\cos^2\theta\right)^3$$
$$32\cos^6\theta-48\cos^4\theta+18\cos^2\theta-1$$

6 b We wish to solve

$32x^6 - 48x^4 + 18x - \frac{3}{2} = 0$

We use the substitution $x = \cos\theta$ so that the equation becomes

$$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - \frac{3}{2} = 0$$

i.e.

$$32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \frac{1}{2}$$

Hence

$$\cos 6\theta = \frac{1}{2}$$

The general solution to this is given by

$6\theta = \pm\frac{\pi}{3} + 2k\pi$ where k is any integer i.e.

$$\theta = \pm\frac{\pi}{18} + \frac{k\pi}{3}$$

Trying both choices of sign and varying k gives the following values of $x = \cos\theta$

$x = \pm 0.985$

$x = \pm 0.342$

$x = \pm 0.643$

But these are 6 distinct solutions and since the polynomial has order 6 there are at most 6 unique solutions hence these are all solutions.

7 a $(\cos\theta + \mathrm{i}\sin\theta)^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$ (de Moivre's Theorem.)

$$\begin{aligned}&= \cos^4\theta + {}^4C_1\cos^3\theta(\mathrm{i}\sin\theta) + {}^4C_2\cos^2\theta(\mathrm{i}\sin\theta)^2 \\ &\quad + {}^4C_3\cos\theta(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4 \\ &= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta + 6\mathrm{i}^2\cos^2\theta\sin^2\theta \\ &\quad + 4\mathrm{i}^3\cos\theta\sin^3\theta + \mathrm{i}^4\sin^4\theta \\ &= \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta\end{aligned}$$

(Binomial expansion.)

Hence,

$\cos 4\theta + \mathrm{i}\sin 4\theta = \cos^4\theta + 4\mathrm{i}\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4\mathrm{i}\cos\theta\sin^3\theta + \sin^4\theta$ **(1)**

Equating the imaginary parts of **(1)** gives:

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (as required)

7 b Equating the real parts of **(1)** gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \qquad \frac{(\sin 4\theta \div \cos^4\theta)}{(\cos 4\theta \div \cos^4\theta)}$$

$$= \frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta} - \frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{\frac{4\cancel{\cos^3\theta}}{\cancel{\cos^3\theta}}\frac{\sin\theta}{\cos\theta} - \frac{4\cancel{\cos\theta}\sin^3\theta}{\cancel{\cos\theta}\cos^3\theta}}{\frac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \frac{6\cancel{\cos^2}\theta\sin^2\theta}{\cancel{\cos^2}\theta\cos^2\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Therefore, $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (as required)

c $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

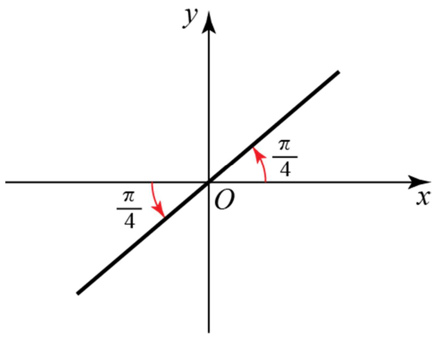
$1 = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$ **(2)**

Let $x = \tan\theta$; then

(2) $\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$

$\tan 4\theta = 1$ ← From part **b**.

$\alpha = \dfrac{\pi}{4}$



$$4\theta = \left\{\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \ldots\right\}$$

$$\theta = \left\{\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \ldots\right\}$$

$\therefore x = \tan\theta = \tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$

$x = 0.19891\ldots, 1.49660\ldots, -5.02733\ldots, -0.66817\ldots,$

$x = 0.20, 1{,}50, -5.03, -0.67$ (2 d.p.)