Complex numbers 1B

1 a $e^{\frac{\pi i}{3}} \times e^{\frac{\pi i}{4}} = e^{\pi i\left(\frac{1}{3}+\frac{1}{4}\right)} = e^{\frac{7\pi i}{12}}$

$= \cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)$

b

$\sqrt{5}e^{i\theta} \times 3e^{3i\theta} = 3\sqrt{5}e^{4i\theta}$

$= 3\sqrt{5}\left(\cos 4\theta + i\sin 4\theta\right)$

c

$\sqrt{2}e^{\frac{2\pi i}{3}} \times e^{-\frac{7\pi i}{3}} \times 3e^{\frac{\pi i}{6}}$

$= 3\sqrt{2}e^{\pi i\left(\frac{2}{3}-\frac{7}{3}+\frac{1}{6}\right)} = 3\sqrt{2}e^{-\frac{9\pi i}{6}}$

$= 3\sqrt{2}\left(\cos\frac{-3\pi}{2} + i\sin\frac{-3\pi}{2}\right) = 3\sqrt{2}i$

2 a

$\frac{2e^{\frac{7\pi i}{2}}}{8e^{\frac{9\pi i}{2}}} = \frac{2}{8}e^{\frac{7\pi i}{2}-\frac{9\pi i}{2}} = \frac{1}{4}e^{-\pi i}$

$= \frac{1}{4}\left(\cos(-\pi) + i\sin(-\pi)\right) = -\frac{1}{4}$

b

$\frac{\sqrt{3}e^{\frac{3\pi i}{7}}}{4e^{-\frac{2\pi i}{7}}} = \frac{\sqrt{3}}{4}e^{\frac{3\pi i}{7}+\frac{2\pi i}{7}} = \frac{\sqrt{3}}{4}e^{\frac{5\pi i}{7}}$

$= \frac{\sqrt{3}}{4}\left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right)$

c

$\frac{\sqrt{2}e^{-\frac{15\pi i}{6}}}{2e^{\frac{\pi i}{3}}} \times \sqrt{2}e^{\frac{19\pi i}{3}} = \frac{\sqrt{2}\times\sqrt{2}}{2}e^{-\frac{15\pi i}{6}-\frac{\pi i}{3}+\frac{19\pi i}{3}}$

$= e^{\frac{21\pi i}{6}} = e^{\frac{7\pi i}{2}} = e^{\frac{3\pi i}{2}} = -i$

3 a

$\left(\cos 2\theta + i\sin 2\theta\right)\left(\cos 3\theta + i\sin 3\theta\right)$

$= e^{2i\theta} \times e^{3i\theta} = e^{5i\theta}$

3 b

$$\left(\cos\frac{3\pi}{11}+i\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11}+i\sin\frac{8\pi}{11}\right)$$
$$=e^{\frac{3\pi i}{11}}\times e^{\frac{8\pi i}{11}}=e^{\pi i}$$

c

$$3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\times 2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$
$$=3e^{\frac{\pi i}{4}}\times 2e^{\frac{\pi i}{12}}=6e^{\frac{3\pi i}{12}+\frac{\pi i}{12}}=6e^{\frac{\pi i}{3}}$$

d

$$\sqrt{6}\left(\cos\frac{-\pi}{12}+i\sin\frac{-\pi}{12}\right)\times\sqrt{3}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$
$$=\sqrt{6}e^{-\frac{\pi i}{12}}\times\sqrt{3}e^{\frac{\pi i}{3}}=\sqrt{18}e^{-\frac{\pi i}{12}+\frac{4\pi i}{12}}=3\sqrt{2}e^{\frac{\pi i}{4}}$$

4 a

$$\frac{\cos 5\theta+i\sin 5\theta}{\cos 2\theta+i\sin 2\theta}=\frac{e^{5i\theta}}{e^{2i\theta}}=e^{5i\theta-2i\theta}=e^{3i\theta}$$

b

$$\frac{\sqrt{2}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=2\sqrt{2}\,\frac{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}}{\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}}=2\sqrt{2}\,\frac{e^{\frac{\pi i}{2}}}{e^{\frac{\pi i}{4}}}$$
$$=2\sqrt{2}e^{\frac{\pi i}{2}-\frac{\pi i}{4}}=2\sqrt{2}e^{\frac{\pi i}{4}}$$

c

$$\frac{3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)}=\frac{3}{4}\times\frac{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}}=\frac{3}{4}\times\frac{e^{\frac{\pi i}{3}}}{e^{\frac{5\pi i}{6}}}$$
$$=\frac{3}{4}e^{\frac{\pi i}{3}-\frac{5\pi i}{6}}=\frac{3}{4}e^{-\frac{\pi i}{2}}$$

5 a We have $z=-9+3\sqrt{3}i$ so that $|z|=\sqrt{81+27}=\sqrt{108}=6\sqrt{3}$

if $\arg z=\theta$ then we have that

$$\tan\theta=-\frac{3\sqrt{3}}{9}=-\frac{\sqrt{3}}{3}$$

Hence

$$\theta=\frac{5\pi}{6}$$

So $z=6\sqrt{3}e^{\frac{5\pi i}{6}}$

5 b We have $|w| = \sqrt{3}$ and $\arg w = \frac{7\pi}{12}$ so by definition we have

$$w = \sqrt{3}\mathrm{e}^{\frac{7\pi i}{12}}$$

c $zw = 6\sqrt{3}\mathrm{e}^{\frac{5\pi i}{6}} \times \sqrt{3}\mathrm{e}^{\frac{7\pi i}{12}} = 18\mathrm{e}^{\frac{17\pi i}{12}} = 18\mathrm{e}^{-\frac{7\pi i}{12}}$

d $\frac{z}{w} = \frac{6\sqrt{3}\mathrm{e}^{\frac{5\pi i}{6}}}{\sqrt{3}\mathrm{e}^{\frac{7\pi i}{12}}} = 6\mathrm{e}^{\frac{5\pi i}{6} - \frac{7\pi i}{12}} = 6\mathrm{e}^{\frac{\pi i}{4}}$

6 We have

$$\frac{(\cos 9\theta + i\sin 9\theta)(\cos 4\theta + i\sin 4\theta)}{\cos 7\theta + i\sin 7\theta} = \frac{\mathrm{e}^{9i\theta} \times \mathrm{e}^{4i\theta}}{\mathrm{e}^{7i\theta}}$$

$$= \mathrm{e}^{9i\theta + 4i\theta - 7i\theta} = \mathrm{e}^{6i\theta} = \cos 6\theta + i\sin 6\theta$$

7 $z = 1 + i\sqrt{3}$ so $|z| = \sqrt{1+3} = 2$ and if $\arg z = \theta$ then we have

$\tan\theta = \sqrt{3}$ so that $\theta = \frac{\pi}{3}$

Now the equation $\left|\frac{z^2}{w}\right| = |z|$ implies that $|w| = |z|$ hence it only remains to find the possible values of $\arg w = \varphi$ we have that

$$\mathrm{Re}\left(\frac{z^2}{w}\right) = 0$$

Which means that z^2w^{-1} is purely imaginary i.e. that $\arg\left(z^2w^{-1}\right) = \pm\frac{\pi}{2}$

So there are two cases to consider, we first consider the case $\arg\left(z^2w^{-1}\right) = \frac{\pi}{2}$

Then if $w = 2\mathrm{e}^{i\varphi}$ we have $\frac{2\pi}{3} - \varphi = \frac{\pi}{2}$ hence $\varphi = \frac{\pi}{6}$ so we have

$$w = 2\mathrm{e}^{\frac{\pi i}{6}}$$

In the second case we have $\frac{2\pi}{3} - \varphi = -\frac{\pi}{2}$ hence $\varphi = \frac{7\pi}{6}$ so we have

$$w = 2\mathrm{e}^{\frac{7\pi i}{6}} = 2\mathrm{e}^{-\frac{5\pi i}{6}}$$

8 a Note that $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4}$ so we can write it in exponential form as

$1 + i = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}}$ hence we have $(1+i)^2 = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}} \times \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}} = 2e^{\frac{i\pi}{2}}$

b We wish to prove by induction that

$$(1+i)^n = 2^{\frac{n}{2}}e^{\frac{in\pi}{4}}$$

Note that the base case is already true for $n = 1$ by the first part of the question so assume the statement is true up to $n = k$ then

$$(1+i)^{k+1} = (1+i) \times 2^{\frac{k}{2}}e^{\frac{ik\pi}{4}} = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}} \times 2^{\frac{k}{2}}e^{\frac{ik\pi}{4}} = 2^{\frac{k+1}{2}}e^{\frac{i(k+1)\pi}{4}}$$

Proving the statement is true for $n = k + 1$ hence the claim is true by induction

8 c $(1+i)^{16} = 2^8 e^{\frac{16\pi i}{2}} = 256$

9 We have

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos-\theta + i\sin-\theta = \cos\theta - i\sin\theta$

Multiplying the two equations gives

$1 = e^{i\theta} \times e^{-i\theta} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$

$= \cos^2\theta + \sin^2\theta$

Challenge

a We want to prove by induction that

$(re^{i\theta})^n = r^n e^{in\theta}$

Clearly the statement is true when $n = 1$, suppose now the statement is true for $n = k$ then we have

$(re^{i\theta})^{k+1} = re^{i\theta} \times (re^{i\theta})^k = re^{i\theta} \times r^k e^{ik\theta} = r^{k+1} e^{i\theta + ik\theta} = r^{k+1} e^{i(k+1)\theta}$

Proving the statement is true for $n = k+1$ hence the claim is true by induction.

b Now we want to show that

$(re^{i\theta})^{-n} = r^{-n} e^{-in\theta}$

Again by definition this is true when $n = 1$ so suppose it is true for $n = k$, then we have

$$(re^{i\theta})^{-(k+1)} = \frac{1}{(re^{i\theta})^{k+1}} = \frac{1}{re^{i\theta} \times (re^{i\theta})^k} = \frac{r^{-k} e^{-ik\theta}}{re^{i\theta}} = r^{-(k+1)} e^{-ik\theta - i\theta}$$

$$= r^{-(k+1)} e^{-i(k+1)\theta}$$

Proving the statement is true for $n = k+1$ hence the claim is true by induction.