Exam-style practice A Level

1 a The normal to the plane is given by the vector product

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -7 & 0\\ 0 & -8 & 4\end{vmatrix}=-28\mathbf{i}-8\mathbf{j}-16\mathbf{k}$$

Hence after dividing the normal by a scalar the vector equation of the plane is

$$\mathbf{r}\cdot(7\mathbf{i}+2\mathbf{j}+4\mathbf{k})=(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot(7\mathbf{i}+2\mathbf{j}+4\mathbf{k})$$
$$\mathbf{r}\cdot(7\mathbf{i}+2\mathbf{j}+4\mathbf{k})=-7+6+8=7$$

So the Cartesian equation is
$7x+2y+4z=7$

b The volume is given by the formula
$$\frac{1}{6}\left|\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC})\right|$$
Which is given by the determinant

$$\frac{1}{6}\begin{vmatrix}-6 & -5 & 0\\ 2 & -7 & 0\\ 0 & -8 & 4\end{vmatrix}=\frac{1}{6}(168+40)=\frac{104}{3}$$

c The line $\overrightarrow{ED}$ can be expressed in terms of unit normal to the plane $\mathbf{n}$ as
$$\left(\overrightarrow{AD}\cdot\mathbf{n}\right)\mathbf{n}$$
By drawing a picture we can see that
$$\sin\theta=\frac{\left|\overrightarrow{ED}\right|}{\left|\overrightarrow{CD}\right|}=\frac{\left|\overrightarrow{AD}\cdot\mathbf{n}\right|}{\left|\overrightarrow{CD}\right|}$$
We have
$$\left|\overrightarrow{CD}\right|=\sqrt{36+9+16}=\sqrt{61}$$
and
$$\left|\overrightarrow{AD}\cdot\mathbf{n}\right|=\frac{|-42-10|}{\sqrt{49+4+16}}=\frac{52}{\sqrt{69}}$$
Hence
$$\sin\theta=\frac{52}{\sqrt{61}\sqrt{69}}$$
So we have $\theta=0.930$

2 a We have the following tableau of values

x_i	0.5	0.75	1	1.25	1.5
y_i	0.390	0.511	0.678	0.867	0.993

Hence using Simpson's rule our approximation is

$$\int_{0.5}^{1.5} f(x)dx$$
$$\approx\frac{1}{3}\times0.25(f(x_0)+4(f(x_1)+f(x_3))+2f(x_2)+f(x_4))$$
$$=0.68778\text{ (5dp)}$$

2 b To solve the integral exactly we use the substitution $t=\tan\frac{x}{2}$ we have

$$\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)=\frac{1}{2}(1+t^2)$$

So the transformed integral becomes

$$\int_{\tan\frac{1}{4}}^{\tan\frac{3}{4}}\frac{2}{4-3\sin x}(1+t^2)^{-1}dt$$

Now recall that $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}$

and hence

$$\sin x=2\tan\frac{x}{2}\cos^2\frac{x}{2}=\frac{2t}{1+t^2}$$

So the transformed integral becomes

$$\int_{\tan\frac{1}{4}}^{\tan\frac{3}{4}}\frac{2(1+t^2)^{-1}}{4-6t(1+t^2)^{-1}}dt=\int_{\tan\frac{1}{4}}^{\tan\frac{3}{4}}\frac{1}{2t^2-3t+2}dt$$

Now complete the square to get

$$\int_{\tan\frac{1}{4}}^{\tan\frac{3}{4}}\frac{1}{\left(\frac{4t-3}{2\sqrt{2}}\right)^2+\frac{7}{8}}dt=\int_{\tan\frac{1}{4}}^{\tan\frac{3}{4}}\frac{8}{(4t-3)^2+7}dt$$

Now use the substitution $u=\frac{4t-3}{\sqrt{7}}$ so the integral becomes

$$\int_{\frac{4\tan\frac{1}{4}-3}{\sqrt{7}}}^{\frac{4\tan\frac{3}{4}-3}{\sqrt{7}}}\frac{8}{7u^2+7}\frac{\sqrt{7}}{4}du=\frac{2}{\sqrt{7}}\int_{\frac{4\tan\frac{1}{4}-3}{\sqrt{7}}}^{\frac{4\tan\frac{3}{4}-3}{\sqrt{7}}}\frac{1}{u^2+1}du$$

$$=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{4\tan\frac{3}{4}-3}{\sqrt{7}}\right)-\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{4\tan\frac{1}{4}-3}{\sqrt{7}}\right)$$

$$=0.68795$$

c The percentage error is given by

$$\frac{|0.68795-0.68778|}{0.68795}=0.02\%$$

The approximation could be improved by taking more intervals

3 a We have to solve the differential equation

$$xy\frac{dy}{dx}+3x^2+y^2=0$$

Let us use the substitution $y=vx$ so that by the product rule

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

So the differential equation becomes

$$x^2v\left(v+x\frac{dv}{dx}\right)+3x^2+v^2x^2=0$$

Cancelling common factors and simplifying leads to

$$vx\frac{dv}{dx}+3+2v^2=0$$

Which can be written as

$$x\frac{dv}{dx}+\frac{3+2v^2}{v}=0$$

b We have

$$\frac{v}{3+2v^2}\frac{dv}{dx}=-\frac{1}{x}$$

Integrating both sides with respect to x yields

$$\int\frac{v}{3+2v^2}\,dv=\int-\frac{1}{x}\,dx$$

Hence

$$\frac{1}{2}\ln\left(3+2v^2\right)=-\ln(x)+C$$

Hence

$$\left(3+2v^2\right)^{\frac{1}{2}}=\frac{A}{x}$$

For some positive constant A

Squaring gives

$$3+2v^2=\frac{A}{x^2}$$

Which simplifies to

$$3x^4+2y^2x^2=A$$

Now we know that when $x=1$ we have $y=5$ so A satisfies

$$A=3+50=53$$

Hence the full solution is

$$3x^4+2y^2x^2=53$$

3 c When $y = 0$ we have

$3x^4 = 53$

So

$x = \sqrt[4]{\frac{53}{3}} = 2.0502$

Hence the cliff is 205.02m tall.

d When x is small the model is approximated by

$2x^2y^2 = 53$

Which suggests the velocity must be very large when x is small which is unrealistic as the velocity should be small after just jumping.

4 a As $x \to 1$ we have

$5x^4 - 3x^2 - 1 \to 1$

but

$11 - 2x - 9x^3 \to 0$

Hence we are not in a situation where L'Hopistal's rule applies.

b Now both terms in the expression tend to zero as $x \to 1$ so we are in a position where we can use L'Hopistal's rule which gives

$$\lim_{x\to1} \frac{5x^4 - 3x^2 - 2}{11 - 2x - 9x^3} = \lim_{x\to1} \frac{20x^3 - 6x}{-2 - 27x^2}$$

And one may evaluate the second limit to get

$$\lim_{x\to1} \frac{20x^3 - 6x}{-2 - 27x^2} = -\frac{14}{29}$$

5 a If the line intersects the hyperbola then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and

$y = mx + c$

Hence

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

So

$$b^2x^2 - a^2\left(m^2x^2 + 2mcx + c^2\right) = a^2b^2$$

Rearranging gives

$$\left(b^2 - a^2m^2\right)x^2 - 2a^2mcx - a^2\left(b^2 + c^2\right) = 0$$

Now the line is a tangent if and only if it intersects in exactly one point, hence the above quadratic must have one root so by considering the discriminant we have

$$4a^4m^2c^2 + 4a^2\left(b^2 - a^2m^2\right)\left(b^2 + c^2\right) = 0$$

Hence

$$a^2m^2c^2 + \left(b^2 - a^2m^2\right)\left(b^2 + c^2\right) = 0$$

So

$$a^2m^2c^2 = -b^4 - b^2c^2 + a^2m^2b^2 + a^2m^2c^2$$

So

$$b^4 + b^2c^2 = a^2m^2b^2$$

Hence the result

$$b^2 + c^2 = a^2m^2$$

5 b We now consider the hyperbola

$$\frac{x^2}{26}-\frac{y^2}{25}=1$$

And we want to find the tangent that passes through $(2,3)$ using the result from the previous part we must have

$$26m^2=25+c^2$$

And since the tangent passes through $(2,3)$ we must have

$$3=2m+c$$

Hence combining we have

$$26m^2=25+(3-2m)^2$$

So

$$26m^2=25+9-12m+4m^2$$

$$22m^2+12m-34=0$$

$$11m^2+6m-17=0$$

Which we can solve to give the solutions

$$m=1$$

and

$$m=-\frac{17}{11}$$

Hence there are two tangents that pass through the prescribed point given by

$$y=x+1$$

and

$$y=-\frac{17}{11}x+\frac{67}{11}$$

6 We want to find a series solution of

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2y=0$$

Under the initial conditions at $x=0$

$$y=\frac{dy}{dx}=1$$

Using the differential equation at $x=0$ we must have

$$\frac{d^2y}{dx^2}+3=0$$

Hence the coefficient of x^2 in the series solution is $-\frac{3}{2}$, differentiating the differential equation gives

$$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+2\frac{dy}{dx}=0$$

Evaluating at $x=0$ leads to

$$\frac{d^3y}{dx^3}-6+2=0$$

Hence the coefficient of x^3 is $\frac{4}{3!}$

So the series solution up to the x^3 term is

$$y=1+x-\frac{3}{2}x^2+\frac{2}{3}x^3$$

7 We wish to solve the inequality

$$\left|\frac{x}{x+3}\right| < 2 - x$$

First of all note that when $x > 2$ the right hand side is negative and so the inequality can never be satisfied.

Now consider the first case where the expression inside the modulus is positive, i.e.

$$\frac{x}{x+3} > 0$$

and note that this is satisfied when either $x > 0$ or $x < -3$ if $x > 0$ then the inequality becomes

$$\frac{x}{x+3} < 2 - x$$

Hence

$$x < (2-x)(x+3)$$

So

$$x^2 + 2x - 6 < 0$$

By finding the roots one sees that this quadratic is negative if and only if

$$-1-\sqrt{7} < x < \sqrt{7} - 1$$

But we are in the case where $x > 0$ hence in this case the inequality is satisfied if

$$0 < x < \sqrt{7} - 1$$

Now consider the case $x < -3$ then the inequality is

$$\frac{x}{x+3} < 2 - x$$

Which becomes

$$x > (2-x)(x+3)$$

Hence

$$x^2 + 2x - 6 > 0$$

Which is satisfied when $x < 1 - \sqrt{7}$

Finally consider the case where the expression inside the modulus is negative in which case we have $-3 < x < 0$ so the inequality becomes

$$-\frac{x}{x+3} < 2 - x$$

So

$$\frac{x}{x+3} > x - 2$$

So

$$x > (x-2)(x+3)$$

Hence

$$x > x^2 + x - 6$$

Hence

$$x^2 < 6 \text{ so } x > -\sqrt{6}$$

Hence the solution set from this case is $-\sqrt{6} < x < 0$ putting all the cases together gives the solution set

$$\left\{x : x < 1 - \sqrt{7}\right\} \bigcup \left\{x : -\sqrt{6} < x < \sqrt{7} - 1\right\}$$

8 We have $y = \mathrm{e}^x \sin x$ we can write $y = uv$ where $u = \mathrm{e}^x$ and $v = \sin x$ then Leibnitz's theorem gives

$$\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} = \sum_{k=0}^{6} \binom{6}{k} \frac{\mathrm{d}^{6-k} u}{\mathrm{d}x^{6-k}} \frac{\mathrm{d}^k v}{\mathrm{d}x^k}$$

Note that

$\frac{\mathrm{d}^k u}{\mathrm{d}x^k} = u = \mathrm{e}^x$ for all k then the sum becomes

$$\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} = \mathrm{e}^x(\sin x + 6\cos x - 15\sin x - 20\cos x$$
$$+ 15\sin x + 6\cos x - \sin x)$$
$$= -8\mathrm{e}^x \cos x$$

And the usual product rule gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x\left(\sin x + \cos x\right)$$

Hence

$$\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + 8\frac{\mathrm{d}y}{\mathrm{d}x} - 8y$$
$$= -8\mathrm{e}^x \cos x + 8\mathrm{e}^x\left(\sin x + \cos x\right) - 8\mathrm{e}^x \sin x$$
$$= 0$$

As required.