Practice paper, AS Level

1 $\frac{1}{x+1} < \frac{x}{x+3}$

$(x+1)(x+3)^2 < x(x+1)^2(x+3)$

$(x+3)(x+1)[x(x+1)-(x+3)] > 0$

$(x+3)(x+1)(x^2-3) > 0$

$f(x) = (x+3)(x+\sqrt{3})(x+1)(x-\sqrt{3}) > 0$

Critical points are $x = -3, -\sqrt{3}, -1, \sqrt{3}$

$x > \sqrt{3} \Rightarrow f(x) > 0$

Similarly for other intervals to give solution

$\{x : x < -3\} \cup \{x : -\sqrt{3} < x < -1\} \cup \{x : x > \sqrt{3}\}$

2 a $t = \tan\frac{x}{2}$

$\cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$

$2\sin x - 5\cos x = 2$

$2\frac{2t}{1+t^2} - 5\frac{1-t^2}{1+t^2} = 2$

$4t - 5 + 5t^2 = 2 + 2t^2$

$3t^2 + 4t - 7 = 0$

b $(3t+7)(t-1) = 0$

$t = -\frac{7}{3}, t = 1$

$\tan\frac{x}{2} = -\frac{7}{3}, \tan\frac{x}{2} = 1$

$0 < x < 2\pi$

$x = 2\arctan\left(-\frac{7}{3}\right) + 2\pi = 3.95$ (2 d.p.)

$x = 2\arctan 1 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

3 $\frac{d^2y}{dx^2} = e^{xy} - \frac{dy}{dx}$

$y_0 = 1, \left(\frac{dy}{dx}\right)_0 = 2, h = 0.1$

$\left(\frac{d^2y}{dx^2}\right)_0 = e^{0\times y_0} - \left(\frac{dy}{dx}\right)_0$

$= e^0 - 2 = -1$

$\left(\frac{dy}{dx}\right)_0 = 2 \approx \frac{y_1 - y_{-1}}{2h}$

$\left(\frac{d^2y}{dx^2}\right)_0 = -1 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$

$y_{-1} = y_1 - 4h = y_1 - 0.4$

$y_1 + y_{-1} = 2 - h^2 = 1.99$

$2y_1 - 0.4 = 1.99, y_1 = 1.195$

4 a $y^2 = 40x$ intersects $x = k$ at P, Q

tangent to curve at P intersects

y-axis at $(0, 2\sqrt{10})$

$P: y = \sqrt{40k} = 2\sqrt{10k}, x = k$

$\frac{dy}{dx} = \frac{20}{y} = \frac{20}{2\sqrt{10k}} = \sqrt{\frac{10}{k}}$

tangent to curve at P:

$y - 2\sqrt{10k} = \sqrt{\frac{10}{k}}(x-k)$

$y = \sqrt{\frac{10}{k}}x + \sqrt{10k}$

$x = 0 \Rightarrow y = \sqrt{10k} = 2\sqrt{10}, k = 4$

b Two tangents intersect at $y = 0$ by symmetry

$y = \frac{1}{2}\sqrt{10}x + 2\sqrt{10}$

$y = 0 \Rightarrow x = -4$

4 c Let A be the point of intersection of the two tangents, $(-4,0)$ and B be the point $(4,0)$

The triangle PAB has area $\frac{8\times 4\sqrt{10}}{2}$

By symmetry, $\frac{R}{2} = \text{Area}_{PAB} - \int_0^4 \sqrt{40x}\,dx$

$\int_0^4 \sqrt{40x}\,dx = \frac{4}{3}\sqrt{10}\left[x^{\frac{3}{2}}\right]_0^4 = \frac{32}{3}\sqrt{10}$

$R = 2\left(16\sqrt{10} - \frac{32}{3}\sqrt{10}\right) = \frac{32}{3}\sqrt{10}$

$= 33.7 \quad \text{(1 d.p.)}$

5 a $\overrightarrow{OA} = 6\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

$\overrightarrow{OB} = -2\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\overrightarrow{OC} = -\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$

$$\overrightarrow{OB}\times\overrightarrow{OC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 2 & 3 \\ -1 & 4 & -1 \end{vmatrix}$$

$= \boldsymbol{i}(-2-12) + \boldsymbol{j}(-3-2) + \boldsymbol{k}(-8+2)$

$= -14\boldsymbol{i} - 5\boldsymbol{j} - 6\boldsymbol{k}$

b $\text{Area}_{OBC} = \frac{1}{2}\left|\overrightarrow{OB}\times\overrightarrow{OC}\right|$

$= \frac{1}{2}\sqrt{(-14)^2 + (-5)^2 + (-6)^2}$

$= \frac{1}{2}\sqrt{257} = 8.02 \quad \text{(2 d.p.)}$

c Volume of dice is $\frac{1}{6}\left|\overrightarrow{OA}.\left(\overrightarrow{OB}\times\overrightarrow{OC}\right)\right|$

$$= \frac{1}{6}\left|\begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix}.\begin{pmatrix} -14 \\ -5 \\ -6 \end{pmatrix}\right| = \frac{1}{6}\left|-84-20-12\right|$$

$= \frac{116}{6} = \frac{58}{3}\text{cm}^3$

Volume of filament available is $\frac{1000\text{g}}{1.35\text{g/cm}^3}$

$\frac{1000\text{g}}{1.35\text{g/cm}^3} \times \frac{1}{\frac{58}{3}\text{cm}^3} = \frac{10000}{263} = 38.02$

Maximum number of dice that can be made is 38

5 d The answer is an overestimate as we are assuming there is no waste.