Review Exercise 2

1 $\sin\left(\frac{x}{2}\right)=\frac{12}{13}$

$\cos\left(\frac{x}{2}\right)=\sqrt{1-\left(\frac{12}{13}\right)^2}=\frac{5}{13}$

$\tan\left(\frac{x}{2}\right)=\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$

$=\frac{12}{13}\div\frac{5}{13}=\frac{12}{5}$

$t=\tan\left(\frac{x}{2}\right)=\frac{12}{5}$

$\cot x=\frac{1-t^2}{2t}=-\frac{119}{120}$

2 a $\sin\theta=\frac{\sqrt{6}+\sqrt{2}}{4}$

$\sin^2\theta=\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2$

$=\frac{6+2+2\sqrt{12}}{16}$

$=\frac{2+\sqrt{3}}{4}$

$\cos^2\theta=1-\sin^2\theta$

$=1-\frac{2+\sqrt{3}}{4}=\frac{2-\sqrt{3}}{4}$

$\tan^2\theta=\frac{\sin^2\theta}{\cos^2\theta}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$

$=\frac{2+\sqrt{3}}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}=\left(2+\sqrt{3}\right)^2$

Since $\frac{\pi}{2}<\theta<\pi$, $\tan\theta$ should be negative.

$\tan\theta=-\sqrt{\left(2+\sqrt{3}\right)^2}=-2-\sqrt{3}$

2 b $t = \tan\theta = -2 - \sqrt{3}$

$$\sin 2\theta = \frac{2t}{1+t^2}$$

$$= \frac{2\left(-2-\sqrt{3}\right)}{1+\left(-2-\sqrt{3}\right)^2}$$

$$= \frac{-4-2\sqrt{3}}{1+4+3+4\sqrt{3}}$$

$$= \frac{-4-2\sqrt{3}}{8+4\sqrt{3}}$$

$$= -\frac{\left(4+2\sqrt{3}\right)}{2\left(4+2\sqrt{3}\right)} = -\frac{1}{2}$$

$$\cos 2\theta = \frac{1-t^2}{1+t^2}$$

$$= \frac{1-\left(-2-\sqrt{3}\right)^2}{1+\left(-2-\sqrt{3}\right)^2}$$

$$= \frac{1-4-3-4\sqrt{3}}{1+4+3+4\sqrt{3}}$$

$$= \frac{-6-4\sqrt{3}}{8+4\sqrt{3}}$$

$$= -\frac{2\left(3+2\sqrt{3}\right)}{4\left(2+\sqrt{3}\right)}$$

$$= -\frac{2\left(3+2\sqrt{3}\right)}{4\left(2+\sqrt{3}\right)} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$= -\frac{6-6-3\sqrt{3}+4\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$$

c $2\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$$\theta = \frac{1}{2} \times \frac{7\pi}{6} = \frac{7\pi}{12}$$

3 a $\sec x = \dfrac{1}{\cos x} = \dfrac{1+t^2}{1-t^2}$

$\tan x = \dfrac{2t}{1-t^2}$

$\sec x + \tan x = \dfrac{1+t^2}{1-t^2} + \dfrac{2t}{1-t^2}$

$= \dfrac{1+t^2+2t}{1-t^2}$

$= \dfrac{(1+t)^2}{(1+t)(1-t)}$

$= \dfrac{1+t}{1-t}$

b

$\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right) = \dfrac{\tan\left(\dfrac{\pi}{4}\right)+\tan\left(\dfrac{x}{2}\right)}{1-\tan\left(\dfrac{\pi}{4}\right)\times\tan\left(\dfrac{x}{2}\right)}$

$= \dfrac{1+t}{1-t} = \sec x + \tan x$

4 $t = \tan\left(\dfrac{\theta}{2}\right)$

$\cos\left(\dfrac{\theta}{2}\right) = \dfrac{1}{\sec\left(\dfrac{\theta}{2}\right)} = \dfrac{1}{\sqrt{1+t^2}}$

$2\cos^2\left(\dfrac{\theta}{2}\right) - 1 = 2\left(\dfrac{1}{\sqrt{1+t^2}}\right)^2 - 1$

$= \dfrac{2-1-t^2}{1+t^2} = \dfrac{1-t^2}{1+t^2} = \cos\theta$

5 a $t = \tan\left(\dfrac{x}{2}\right)$

$3\cos x - 4\sin x = 4$

$3\dfrac{1-t^2}{1+t^2} - 4\dfrac{2t}{1+t^2} = 4$

$\dfrac{3-3t^2-8t}{1+t^2} = 4$

$3-3t^2-8t = 4+4t^2$

$4t^2+4-3+3t^2+8t = 0$

$7t^2+8t+1 = 0$

5 b $7t^2 + 8t + 1 = 0$

$(7t+1)(t+1) = 0$

Therefore, $\tan\left(\frac{x}{2}\right) = -\frac{1}{7}, -1$

$\tan\left(\frac{x}{2}\right) = -\frac{1}{7},\ 0 < x < 2\pi$

$\frac{x}{2} = \tan^{-1}\left(-\frac{1}{7}\right) = 2.9997\ldots$

$x = 6.00$ (2 d.p.)

$\tan\left(\frac{x}{2}\right) = -1$

$\frac{x}{2} = \tan^{-1}(-1) = 2.35619\ldots = \frac{3\pi}{4}$

$x = 4.71$ (2 d.p.)

6 a $t = \tan\left(\frac{x}{2}\right)$

$2\sin x + \cos x = 1$

$2\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$

$\frac{4t+1-t^2}{1+t^2} = 1$

$4t + 1 - t^2 = 1 + t^2$

$t^2 + 1 - 4t - 1 + t^2 = 0$

$2t^2 - 4t = 0$

$t^2 - 2t = 0$

b $t^2 - 2t = 0$

$t(t-2) = 0$

Therefore, $\tan\left(\frac{x}{2}\right) = 0, 2$

$\tan\left(\frac{x}{2}\right) = 0,\ 0 \leqslant x \leqslant 2\pi$

$\frac{x}{2} = \tan^{-1} 0 = 0, \pi$

$x = 0, 2\pi$

$\tan\left(\frac{x}{2}\right) = 2$

$\frac{x}{2} = \tan^{-1} 2 = 1.10715\ldots$

$x = 2.21$ (2 d.p.)

7 a Differentiating, we have

$$v = \frac{\mathrm{d}s}{\mathrm{d}x} = 8\cos 4x + 8\cos 2x$$

Using $t = \tan x$ and substituting the t-formulae and using a double-angle formula:

$$\cos 4x = \cos^2 2x - \sin^2 2x$$

$$v = 8\left(\frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2} - \frac{4t^2}{\left(1+t^2\right)^2}\right) + 8\frac{1-t^2}{1+t^2}$$

$$= \frac{8}{\left(1+t^2\right)^2}\left(1-2t^2+t^4-4t^2+1-t^4\right)$$

$$= \frac{16}{\left(1+t^2\right)^2}\left(1-3t^2\right)$$

b Solving for $\frac{\mathrm{d}s}{\mathrm{d}x} = 0$ we have

$t = \pm\frac{1}{\sqrt{3}}$ which implies that

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

To check which point is minimum we differentiate again.

$$\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} = -32\sin 4x - 16\sin 2x$$

$$= -\frac{32}{\left(1+t^2\right)^2}\left(4t\left(1-t^2\right) - t\left(1+t^2\right)\right)$$

$$= -\frac{32}{\left(1+t^2\right)^2}\left(3-5t^2\right)$$

We find $\frac{\mathrm{d}^2 s}{\mathrm{d}x^2}\left(\frac{1}{\sqrt{3}}\right) < 0$ and $\frac{\mathrm{d}^2 s}{\mathrm{d}x^2}\left(-\frac{1}{\sqrt{3}}\right) > 0$

So the minimum is at $t = -\frac{1}{\sqrt{3}}$

Therefore, $\tan x = -\frac{1}{\sqrt{3}} \Rightarrow x = \frac{5\pi}{6}$

$$s = 2\sin\left(\frac{20\pi}{6}\right) + 4\sin\left(\frac{10\pi}{6}\right) + 1$$

$$= -4.196 \text{ m (3 d.p.)}$$

8 a $t = \tan\left(\frac{x}{8}\right)$

$$f'(x) = 5\cos\left(\frac{x}{2}\right) + \frac{11}{4}\cos\left(\frac{x}{4}\right) - 5\sin\left(\frac{x}{4}\right)$$

$$= 5\cos^2\left(\frac{x}{4}\right) - 5\sin^2\left(\frac{x}{4}\right) + \frac{11}{4}\cos\left(\frac{x}{4}\right) - 5\sin\left(\frac{x}{4}\right)$$

$$= 5\left(\frac{1-t^2}{1+t^2}\right)^2 - 5\left(\frac{2t}{1+t^2}\right)^2 + \frac{11}{4}\left(\frac{1-t^2}{1+t^2}\right) - 5\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{20 - 40t^2 + 20t^4 - 80t^2 + 11 - 11t^4 - 40t - 40t^3}{4\left(1+t^2\right)^2}$$

$$= \frac{9t^4 - 40t^3 - 120t^2 - 40t + 31}{4\left(1+t^2\right)^2}$$

$$= \frac{(t+1)\left(9t^3 - 49t^2 - 71t + 31\right)}{4\left(1+t^2\right)^2}$$

b $f'(x) = \frac{(t+1)\left(9t^3 - 49t^2 - 71t + 31\right)}{4\left(1+t^2\right)^2}$

At a stationary point,

$f'(x) = 0 \Rightarrow t = -1 \Rightarrow \frac{x}{8} = \tan^{-1}(-1)$

$\frac{x}{8} = \frac{3\pi}{4} \Rightarrow x = 6\pi$

c From the graph of $y = f(x)$, the maximum value of $f(x)$ is in the range $[60, 62.5]$

If the maximum value of L is 300 litres, then using $L = kf(x)$, k lies in the range $[4.8, 5]$

d The point described by $x = 6\pi$ is the second lowest trough on the left in the graph of $y = f(x)$.

At this point,

$f(x) = 30 + 10\sin 3\pi + 11\sin\frac{3\pi}{2} + 20\cos\frac{3\pi}{2} = 19$

Therefore, the amount of water pumped $L = kf(x)$ lies in the range $[91.2, 95]$

9 a Let $f(x) = \cos 2x$ $\qquad f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -2\sin 2x \qquad f'\left(\frac{\pi}{4}\right) = -2\sin\frac{\pi}{2} = -2$

$f''(x) = -4\cos 2x \qquad f''\left(\frac{\pi}{4}\right) = -4\cos\frac{\pi}{2} = 0$

$f'''(x) = 8\sin 2x \qquad f'''\left(\frac{\pi}{4}\right) = 8\sin\frac{\pi}{2} = 8$

$f^{(iv)}(x) = 16\cos 2x \qquad f^{(iv)}\left(\frac{\pi}{4}\right) = 16\cos\frac{\pi}{2} = 0$

$f^{(v)}(x) = -32\sin 2x \qquad f^{(v)}\left(\frac{\pi}{4}\right) = -32\sin\frac{\pi}{2} = -32$

Taylor's and Maclaurin's series need repeated differentiation and substitution. You need to display these in systematic form, both to help you substitute correctly and to show your working clearly so that the examiner can award you marks.

$f^{(iv)}(x)$ and $f^{(v)}(x)$ are symbols which can be used for the fourth and fifth derivatives of $f(x)$ respectively.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(iv)}(a) + \frac{(x-a)^5}{5!}f^{(v)}(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \cos 2x$ and $a = \frac{\pi}{4}$

$$\cos 2x = \left(x - \frac{\pi}{4}\right) \times (-2) + \frac{\left(x - \frac{\pi}{4}\right)^3}{6} \times 8 + \frac{\left(x - \frac{\pi}{4}\right)^5}{120} \times (-32) + \ldots$$

$$= -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5 + \ldots$$

All of the even derivatives are zero at $x = \frac{\pi}{4}$

b Let $x = 1$, then $x - \frac{\pi}{4} = 0.2146\ldots$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

Substituting into the result of part **a**

$$\cos 2 = -2(0.2146\ldots) + \frac{4}{3}(0.2146\ldots)^3 - \frac{4}{15}(0.2146\ldots)^5 + \ldots$$

$$\approx -0.416147 \text{ (6 d.p.)}$$

This is a very accurate estimate and is correct to 6 decimal places.

10 a Let $f(x) = \ln(\sin x)$ $\quad f\left(\frac{\pi}{6}\right) = \ln\frac{1}{2} = -\ln 2$

$f'(x) = \frac{\cos x}{\sin x} = \cot x$ $\quad f'\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}$

$f''(x) = -\operatorname{cosec}^2 x$ $\quad f''\left(\frac{\pi}{6}\right) = -4$

$\operatorname{cosec}\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$

$f'''(x) = 2\operatorname{cosec}^2 x \cot x$ $\quad f'''\left(\frac{\pi}{6}\right) = 2\times 2^2 \times\sqrt{3} = 8\sqrt{3}$

Using the chain rule,
$\frac{d}{dx}(-\operatorname{cosec}^2 x) = -2\operatorname{cosec} x\frac{d}{dx}(\operatorname{cosec} x)$
$= -2\operatorname{cosec} x \times -\operatorname{cosec} x \cot x$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \ln(\sin x)$ and $a = \frac{\pi}{6}$

$$\ln(\sin x) = -\ln 2 + \left(x - \frac{\pi}{6}\right)\times\sqrt{3} + \frac{1}{2}\left(x-\frac{\pi}{6}\right)^2\times(-4) + \frac{1}{6}\left(x-\frac{\pi}{6}\right)^3\times 8\sqrt{3} + \ldots$$

$$= -\ln 2 + \sqrt{3}\left(x-\frac{\pi}{6}\right) - 2\left(x-\frac{\pi}{6}\right)^2 + \frac{4\sqrt{3}}{3}\left(x-\frac{\pi}{6}\right)^3 + \ldots$$

b Let $x = 0.5$, then $x - \frac{\pi}{6} = -0.0235987\ldots$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

Substituting into the result of part **a**

$$\ln(\sin 0.5) = -\ln 2 + \sqrt{3}(-0.023598\ldots) - 2(-0.023598\ldots)^2 + \frac{4\sqrt{3}}{3}(-0.023\,598\ldots)^3 + \ldots$$

$$\approx -0.735166 \text{ (6 d.p.)}$$

11 a $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x \frac{d}{dx}(\sec x) = 2\sec x \times \sec x \tan x$$
$$= 2\sec^2 x \tan x$$

Using the chain rule for differentiation.

$$\frac{d^3y}{dx^3} = \tan x \frac{d}{dx}(2\sec^2 x) + 2\sec^2 x \frac{d}{dx}(\tan x)$$
$$= 4\sec^2 x \tan^2 x + 2\sec^4 x$$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 2\sec^2 x$ and $v = \tan x$

b Let $y = f(x) = \tan x$

$$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

Using the results in part **a**

$$f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$$

$$f''(x) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 2\times(\sqrt{2})^2\times 1 = 4$$

$$f'''\left(\frac{\pi}{4}\right) = 2\sec^2\frac{\pi}{4}\tan^2\frac{\pi}{4} + 2\sec^4\frac{\pi}{4}$$
$$= 4(\sqrt{2})^2\times 1^2 + 2(\sqrt{2})^4 = 8 + 8 = 16$$

$\sec\frac{\pi}{4} = \sqrt{2}$ and $\tan\frac{\pi}{4} = 1$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is the first four terms of Taylor's series.

Substituting $f(x) = \tan x$ and $x = \frac{\pi}{4}$

You are expanding $\tan x$ about the point $x = \frac{\pi}{4}$, using Taylor's series.

$$\tan x = 1 + \left(x - \frac{\pi}{4}\right)\times 2 + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 \times 4 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 \times 16 + \ldots$$
$$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots$$

11 c Let $x=\frac{3\pi}{10}$, then $x-\frac{\pi}{4}=\frac{3\pi}{10}-\frac{\pi}{4}=\frac{\pi}{20}$

Substituting into the result in part **b**

$$\tan\frac{3\pi}{10}=1+2\left(\frac{\pi}{20}\right)+2\left(\frac{\pi}{20}\right)^2+\frac{8}{3}\left(\frac{\pi}{20}\right)^3+\ldots$$

$$\approx 1+\frac{\pi}{10}+\frac{\pi^2}{200}+\frac{\pi^3}{3000}, \text{ as required.}$$

12 a $f(x)=\ln x,\ f'(x)=\frac{1}{x},\ f''(x)=-\frac{1}{x^2}$

$f'''(x)=\frac{2}{x^3}$

$$\ln x=f(1)+f'(1)(x-1)+(x-1)^2\frac{f''(1)}{2!}+$$

$$(x-1)^3\frac{f'''(1)}{3!}+\ldots(x-1)^3\frac{f'''(1)}{3!}+\ldots$$

$$=0+(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\ldots$$

b $\lim\limits_{x\to1}\frac{2\ln x}{x^2-3x+2}$

$$=\lim_{x\to1}\frac{2(x-1)\left(1-\frac{1}{2}(x-1)+\frac{1}{3}(x-1)^2\right)}{(x-1)(x-2)}$$

$$=\lim_{x\to1}\frac{2\left(1-\frac{1}{2}(x-1)+\frac{1}{3}(x-1)^2\right)}{(x-2)}$$

$$=\frac{2(1-0+0)}{1-2}=-2$$

13 a $f(x)=\sinh x,\ f'(x)=\cosh x$

$f''(x)=\sinh x,\ f'''(x)=\cosh x,\ldots$

$$\sinh x=f(0)+(x-0)f'(0)+(x-0)^2\frac{f''(0)}{2!}+(x-0)^3\frac{f'''(0)}{3!}+(x-0)^4\frac{f''''(0)}{4!}+(x-0)^5\frac{f'''''(0)}{5!}+\ldots$$

$$\sinh x=x+\frac{1}{6}x^3+\frac{1}{120}x^5+\ldots$$

13 b Similarly,

$\cosh x = 1 + \frac{1}{4}x^2 + \frac{1}{24}x^4 + \dots$

$$\lim_{x\to 0}\left(x\operatorname{cosech}(2x)\right) = \lim_{x\to 0}\frac{x}{2\sinh x\cosh x}$$

$$= \lim_{x\to 0}\frac{x}{2\left(x+\frac{1}{6}x^3+\frac{1}{120}x^5+\dots\right)\left(1+\frac{1}{4}x^2+\frac{1}{24}x^4+\dots\right)}$$

$$= \lim_{x\to 0}\frac{1}{2\left(1+\frac{1}{6}x^2+\frac{1}{120}x^4+\dots\right)\left(1+\frac{1}{4}x^2+\frac{1}{24}x^4+\dots\right)}$$

$$= \frac{1}{2\times 1\times 1} = \frac{1}{2}$$

14 a $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$ **(1)**

Differentiate **(1)** throughout with respect to x

$$-2x\frac{d^2y}{dx^2} + (1-x^2)\frac{d^3y}{dx^3} - \frac{dy}{dx} - x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \quad \textbf{(2)}$$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 1-x^2$ and $v = \frac{d^2y}{dx^2}$, $\frac{d}{dx}\left((1-x^2)\frac{d^2y}{dx^2}\right)$

$= \frac{d^2y}{dx^2}\frac{d}{dx}(1-x^2) + (1-x^2)\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

$= \frac{d^2y}{dx^2}\times -2x + (1-x^2)\frac{d^3y}{dx^3}$

Substituting $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$ into **(2)**

$$0 + \frac{d^3y}{dx^3} + 1 - 0 - 2 = 0$$

At $x = 0$, $\frac{d^3y}{dx^3} = 1$

14 b Let $y = \mathrm{f}(x)$

From the data in the question

$\mathrm{f}(0) = 2,\ \mathrm{f}'(0) = -1$

At $x = 0$, **(1)** above becomes

$\mathrm{f}''(0) + 2 \times 2 = 0 \Rightarrow \mathrm{f}''(0) = -4$

And the result to part **a** becomes

$\mathrm{f}'''(0) = 1$

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3

$$y = 2 + x \times (-1) + \frac{x^2}{2} \times (-4) + \frac{x^3}{6} \times 1 + \ldots$$
$$= 2 - x - 2x^2 + \frac{1}{6}x^3 + \ldots$$

15 a $(1+2x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = x + 4y^2$ *

You need to differentiate $4y^2$ implicitly with respect to x
$\dfrac{\mathrm{d}}{\mathrm{d}x}(4y^2) = \dfrac{\mathrm{d}y}{\mathrm{d}x} \times \dfrac{\mathrm{d}}{\mathrm{d}y}(4y^2) = 8y\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Differentiate * throughout with respect to x

$$2\frac{\mathrm{d}y}{\mathrm{d}x} + (1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 8y\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 8y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$= 1 + 2(4y-1)\frac{\mathrm{d}y}{\mathrm{d}x} \quad \textbf{(1)} \text{ as required.}$$

b Differentiate **(1)** throughout with respect to x

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2(4y-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \ldots \quad \textbf{(2)}$$

When using the product rule for differentiation $\dfrac{\mathrm{d}}{\mathrm{d}x}(uv) = v\dfrac{\mathrm{d}u}{\mathrm{d}x} + u\dfrac{\mathrm{d}v}{\mathrm{d}x}$ with $u = 2(4y-1)$ and $v = \dfrac{\mathrm{d}y}{\mathrm{d}x}$, $2(4y-1)$ must be differentiated implicitly with respect to x. So $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2(4y-1)\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$

$$= 8\frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}y}{\mathrm{d}x} + 2(4y-1)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$
$$= 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2(4y-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

15 c Let $y = \mathrm{f}(x)$

From the data in the question

$$\mathrm{f}(0) = \frac{1}{2}$$

At $x = 0,\ y = \frac{1}{2}$, $*$ becomes

$$\mathrm{f}'(0) = 4 \times \left(\frac{1}{2}\right)^2 = 1$$

At $x = 0,\ y = \frac{1}{2},\ \frac{\mathrm{d}y}{\mathrm{d}x} = 1$, **(1)** becomes

$$\mathrm{f}''(0) = 1 + 2\left(4 \times \frac{1}{2} - 1\right) \times 1 = 3$$

At $x = 0,\ y = \frac{1}{2},\ \frac{\mathrm{d}y}{\mathrm{d}x} = 1,\ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 3$, **(2)** becomes

$$2 \times 3 + \mathrm{f}'''(0) = 8 \times 1^2 + 2\left(4 \times \frac{1}{2} - 1\right) \times 3$$

$$6 + \mathrm{f}'''(0) = 8 + 6 \Rightarrow \mathrm{f}'''(0) = 8$$

$$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3

$$y = \frac{1}{2} + x \times 1 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 8 + \ldots$$
$$= \frac{1}{2} + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \ldots$$

16 a Let $y = f(x)$

From the data in the question

$f(0) = 1$

$$\frac{dy}{dx} = y^2 + xy + x \quad \textbf{(1)}$$

y^2 has to be differentiated implicitly by x. So

$$\frac{d}{dx}(y^2) = \frac{dy}{dx} \times \frac{d}{dy}(y^2) = \frac{dy}{dx} \times 2y$$

At $x = 0$, $y = 1$, **(1)** becomes

$f'(0) = 1^2 + 0 + 0 = 1$

Differentiating **(1)** throughout by x

$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + y + x\frac{dy}{dx} + 1 \quad \textbf{(2)}$$

Using the product rule for differentiation

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$ with

$u = x$ and $v = y$,

$$\frac{d}{dx}(xy) = y\frac{dx}{dx} + x\frac{dy}{dx} = y \times 1 + x\frac{dy}{dx}$$

At $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, **(2)** becomes

$f''(0) = 2 \times 1 \times 1 + 1 + 0 + 1 = 4$

Differentiate **(2)** throughout by x

$$\frac{d^3y}{dx^3} = 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} \quad \textbf{(3)}$$

Using the product rule for differentiation

$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 2y$ and $v = \frac{dy}{dx}$,

$$\frac{d}{dx}\left(2y\frac{dy}{dx}\right) = \frac{dy}{dx}\frac{d}{dx}(2y) + 2y\frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{dy}{dx} \times 2\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}$$

At $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 4$, **(3)** becomes

$f'''(0) = 2 \times 1^2 + 2 \times 1 \times 4 + 1 + 1 + 0 = 12$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$y = 1 + x \times 1 + \frac{x^2}{2} \times 4 + \frac{x^3}{6} \times 12 + \ldots$$
$$= 1 + x + 2x^2 + 2x^3 + \ldots$$

b At 0.1

$y = 1 + 0.1 + 2(0.1)^2 + 2(0.1)^3 + \ldots$

$\approx 1 + 0.1 + 0.02 + 0.002 = 1.122$

$y \approx 1.12$ (2 d.p.)

17 a Rearranging the differential equation in the question

$$(y^2+y)\frac{dy}{dx}=x+3 \qquad \textbf{(1)}$$

The right hand side of the equation in the question would be hard to repeatedly differentiate as a quotient, so multiply both sides by $y+1$

Let $y=f(x)$
From the data in the question
$f(0)=1.5$
At $x=0,\ y=1.5$, **(1)** becomes

$$(1.5^2+1.5)f'(0)=0+3 \Rightarrow f'(0)=\frac{3}{3.75}=0.8$$

Differentiate **(1)** throughout by x

$$(2y+1)\left(\frac{dy}{dx}\right)^2+(y^2+y)\frac{d^2y}{dx^2}=1 \qquad \textbf{(2)}$$

At $x=0,\ y=1.5,\ \frac{dy}{dx}=0.8$, **(2)** becomes

$$4\times0.8^2+(1.5^2+1.5)f''(0)=1$$
$$f''(0)=\frac{1-4\times0.8^2}{3.75}=-0.416$$

Differentiate **(2)** throughout by x

Differentiating $\left(\frac{dy}{dx}\right)^2$ by x, using the chain rule

$$\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right)=2\frac{dy}{dx}\times\frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$=2\frac{dy}{dx}\times\frac{d^2y}{dx^2}$$

$$2\left(\frac{dy}{dx}\right)^3+(2y+1)2\times\frac{dy}{dx}\times\frac{d^2y}{dx^2}+(2y+1)\frac{dy}{dx}\times\frac{d^2y}{dx^2}+(y^2+y)\frac{d^3y}{dx^3}=0$$
$$2\left(\frac{dy}{dx}\right)^3+3(2y+1)\frac{dy}{dx}\frac{d^2y}{dx^2}+(y^2+y)\frac{d^3y}{dx^3}=0 \qquad \textbf{(3)}$$

At $x=0,\ y=1.5,\ \frac{dy}{dx}=0.8,\ \frac{d^2y}{dx^2}=-0.416$, **(2)** becomes

$$2\times0.8^3+3\times4\times0.8\times-0.416+(1.5^2+1.5)f'''(0)=0$$
$$1.204-3.9936+3.75f'''(0)=0$$
$$f'''(0)=\frac{3.9936-1.204}{3.75}=0.79189\dot{3}$$

This is a recurring decimal. There is an exact fraction $\frac{7424}{9375}$

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$$
$$y=1.5+x\times0.8+\frac{x^2}{2}\times-0.416+\frac{x^3}{6}\times0.79189\dot{3}+\ldots$$
$$=1.5+0.8x-0.208x^2+0.13198\dot{2}x^3+\ldots$$

b At x = 1, $y=1.5+0.8(0.1)-0.208(0.1)^2+0.13198\dot{2}(0.1)^3+\ldots=1.578\ldots$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = y$ and $v = \frac{d^2y}{dx^2}$,

$$\frac{d}{dx}\left(y\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dx^2}\times\frac{dy}{dx} + y\times\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{dy}{dx}\times\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3}$$

18 a $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$ **(1)**

Differentiate **(1)** throughout with respect to x

$$\frac{dy}{dx}\times\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\times\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$y\frac{d^3y}{dx^3} = -3\frac{dy}{dx}\frac{d^2y}{dx^2} - \frac{dy}{dx} = -\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right)$$

$$\frac{d^3y}{dx^3} = -\frac{1}{y}\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right) \quad \textbf{(2)}$$

The wording of the question requires you to make $\frac{d^3y}{dx^3}$ the subject of the formula. There are many possible alternative forms for the answer.

b Let $y = f(x)$
From the data in the question

$f(0) = 1,\ f'(0) = 1$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1$, **(1)** becomes

$1 \times f''(0) + 1^2 + 1 = 0 \Rightarrow f''(0) = -2$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = -2$, **(2)** becomes

$$f'''(0) = -\frac{1}{1}\times 1(3\times -2 + 1) = -1(-6+1) = 5$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$y = 1 + x\times 1 + \frac{x^2}{2}\times -2 + \frac{x^3}{6}\times 5 + \ldots$$

$$= 1 + x - x^2 + \frac{5}{6}x^3 + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3

c The series expansion up to and including the term in x^3 can be used to estimate y if x is small. So it would be sensible to use it at $x = 0.2$ but not at $x = 50$

19 a $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y^2 = 6$ **(1)**

Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 0$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 0$, **(1)** becomes

$f''(0) - 4\times 0 + 3\times 1^2 = 6 \Rightarrow f''(0) = 3$

Differentiate **(1)** throughout with respect to x

$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 6y\frac{dy}{dx} = 0$ **(2)**

$3y^2$ has to be differentiated implicitly with respect to x

So $\frac{d}{dx}(3y^2) = \frac{dy}{dx}\times\frac{d}{dy}(3y^2) = \frac{dy}{dx}\times 6y$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 3$, **(3)** becomes

$f'''(0) - 4\times 3 + 6\times 1\times 0 = 0 \Rightarrow f'''(0) = 12$

Differentiate **(2)** throughout with respect to x

$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2} = 0$ **(3)**

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

with $u = 6y$ and $v = \frac{dy}{dx'}$

$\frac{d}{dx}\left(6y\frac{dy}{dx}\right)$

$= \frac{dy}{dx}\frac{d}{dy}(6y) + 6y\frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2}$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 3,\ \frac{d^3y}{dx^3} = 12,$

(3) becomes

$f^{(iv)}(0) - 4\times 12 + 6\times 0^2 + 6\times 1\times 3 = 0$

$f^{(iv)}(0) = 48 - 18 = 30$

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \ldots$

$y = 1 + x\times 0 + \frac{x^2}{2}\times 3 + \frac{x^3}{6}\times 12 + \frac{x^4}{24}\times 30 + \ldots$

$= 1 + \frac{3}{2}x^2 + 2x^3 + \frac{5}{4}x^4 + \ldots$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4

19 b At $x = 0.2$

$y = 1 + 0.06 + 0.016 + 0.002 + \ldots \approx 1.078$

$y \approx 1.08\,(2\text{ d.p.})$

20 Let $u = x^3$ and $v = e^{3x}$

$\frac{du}{dx} = 3x^2$, $\frac{d^2u}{dx^2} = 6x$, $\frac{d^3u}{dx^3} = 6$

The subsequent derivatives of u are all zero.

$$\frac{d^n v}{dx^n} = 3^n e^{3x}$$

Using Leibnitz's theorem,

$$\frac{d^n y}{dx^n} = u\frac{d^n v}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \ldots +$$
$$\frac{n(n-1)}{2}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \frac{n(n-1)(n-2)}{6}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}}$$
$$= x^3(3^n e^{3x}) + n(3x^2)(3^{n-1}e^{3x}) + \frac{n(n-1)}{2}(6x)(3^{n-2}e^{3x}) + \frac{n(n-1)(n-2)}{6}(6)(3^{n-3}e^{3x})$$
$$= 3^{n-3}e^{3x}(27x^3 + 27nx^2 + 9n(n-1)x + n(n-1)(n-2))$$

21 Let $u = e^x$ and $v = \sin x$

$$\frac{d^n u}{dx^n} = e^x$$

$\frac{dv}{dx} = \cos x$, $\frac{d^2v}{dx^2} = -\sin x$, $\frac{d^3v}{dx^3} = -\cos x$

$\frac{d^4v}{dx^4} = \sin x$, $\frac{d^5v}{dx^5} = \cos x$, $\frac{d^6v}{dx^6} = -\sin x$

Using Leibnitz's theorem,

$$\frac{d^6 y}{dx^6} = u\frac{d^6 v}{dx^6} + 6\frac{du}{dx}\frac{d^5 v}{dx^5} + 15\frac{d^2u}{dx^2}\frac{d^4v}{dx^4} +$$
$$20\frac{d^3u}{dx^3}\frac{d^3v}{dx^3} + 15\frac{d^4u}{dx^4}\frac{d^2v}{dx^2} + 6\frac{d^5u}{dx^5}\frac{dv}{dx} + \frac{d^6u}{dx^6}v$$
$$= -e^x\sin x + 6e^x\cos x + 15e^x\sin x - 20e^x\cos x$$
$$-15e^x\sin x + 6e^x\cos x + e^x\sin x$$
$$= -8e^x\cos x$$

$$\frac{d^6y}{dx^6} + 8\frac{dy}{dx} - 8y$$
$$= -8e^x\cos x + 8(e^x\cos x + e^x\sin x) - 8e^x\sin x$$
$$= 0$$

22 Let $f(x) = \ln x$ and $g(x) = x^2 - 1$

$f(1) = \ln 1 = 0$ and $g(1) = 1 - 1 = 0$, so we can apply L'Hospital's rule.

$f'(x) = \frac{1}{x}$ and $g'(x) = 2x$

By L'Hospital's rule,

$$\lim_{x\to 1}\left(\frac{\ln x}{x^2 - 1}\right) = \lim_{x\to 1}\left(\frac{1}{2x^2}\right) = \frac{1}{2}$$

23 Let $f(x) = \ln x$ and $g(x) = \frac{1}{x}$ so that $\lim_{x\to 0}(x\ln x) = \lim_{x\to 0}\left(\frac{\ln x}{\frac{1}{x}}\right)$

$f(0) = \ln 0 = \infty$ and $g(0) = \frac{1}{0} = \infty$, so we can apply L'Hospital's rule.

$f'(x) = \frac{1}{x}$ and $g'(x) = -\frac{1}{x^2}$

By L'Hospital's rule,

$$\lim_{x\to 0}(x\ln x) = \lim_{x\to 0}\left(\frac{\ln x}{\frac{1}{x}}\right) = \lim_{x\to 0}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)$$

$$= \lim_{x\to 0}(-x) = 0$$

24 Let $f(x) = xe^x$ and $g(x) = 2\sin x$

$f(0) = 0 \times e^0 = 0$ and $g(0) = \sin 0 = 0$, so we can apply L'Hospital's rule.

$f'(x) = e^x + xe^x$ and $g'(x) = 2\cos x$

By L'Hospital's rule, $\lim_{x\to 0}\left(\frac{xe^x}{2\sin x}\right) = \lim_{x\to 0}\left(\frac{e^x(x+1)}{2\cos x}\right) = \frac{1(0+1)}{2} = \frac{1}{2}$

25 Let $f(x) = e^x - \cos x$ and $g(x) = x$

$f(0) = e^0 - 1 = 0$ and $g(0) = 0$, so we can apply L'Hospital's rule.

$f'(x) = e^x + \sin x$ and $g'(x) = 1$

By L'Hospital's rule,

$$\lim_{x\to 0}\left(\frac{e^x - \cos x}{x}\right) = \lim_{x\to 0}\left(\frac{e^x + \sin x}{1}\right) = 1$$

26 a $t=\tan\left(\frac{x}{2}\right)$

$\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

$$\frac{1}{1-\sin x+\cos x}$$

$$=\frac{1}{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$=\frac{1+t^2}{1+t^2-2t+1-t^2}$$

$$=\frac{1+t^2}{2(1-t)}$$

$$t=\tan\left(\frac{x}{2}\right)\Rightarrow \mathrm{d}t=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)\mathrm{d}x$$

$$\Rightarrow \mathrm{d}x=\frac{2\mathrm{d}t}{1+t^2}$$

$$\int\frac{1}{1-\sin x+\cos x}\mathrm{d}x$$

$$=\int\frac{1+t^2}{2(1-t)}\frac{2\mathrm{d}t}{1+t^2}=\int\frac{1}{1-t}\mathrm{d}t$$

b $\displaystyle\int_0^{\frac{\pi}{4}}\frac{1}{1-\sin x+\cos x}\mathrm{d}x=\int_0^{\tan\left(\frac{\pi}{8}\right)}\frac{1}{1-t}\mathrm{d}t$

$$=\left[-\ln(1-t)\right]_0^{\tan\left(\frac{\pi}{8}\right)}=0.535\,(3\text{ d.p.})$$

27 $t = \tan\left(\frac{x}{2}\right)$

$\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$\frac{1}{3\sin x - 4\cos x}$$
$$= \frac{1}{3\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}}$$
$$= \frac{1+t^2}{6t - 4 + 4t^2}$$
$$= \frac{1+t^2}{2(2t-1)(t+2)}$$

$t = \tan\left(\frac{x}{2}\right) \Rightarrow \mathrm{d}t = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\mathrm{d}x$

$\Rightarrow \mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$

$$\int \frac{1}{3\sin x - 4\cos x}\mathrm{d}x$$
$$= \int \frac{1}{(2t-1)(t+2)}\mathrm{d}t$$
$$= \frac{1}{5}\int\left(\frac{2}{2t-1} - \frac{1}{t+2}\right)\mathrm{d}t$$
$$= \frac{1}{5}\left(\ln\left(2\tan\left(\frac{x}{2}\right)-1\right) - \ln\left(\tan\left(\frac{x}{2}\right)+2\right)\right)$$

$$\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} \frac{1}{3\sin x - 4\cos x}\mathrm{d}x$$
$$= \frac{1}{5}\left[\left(\ln\left(2\tan\left(\frac{x}{2}\right)-1\right) - \ln\left(\tan\left(\frac{x}{2}\right)+2\right)\right)\right]_{\frac{\pi}{2}}^{\frac{7\pi}{6}}$$
$$= \frac{1}{5}\left[\ln\left(\frac{2\tan\left(\frac{x}{2}\right)-1}{\tan\left(\frac{x}{2}\right)+2}\right)\right]_{\frac{\pi}{2}}^{\frac{7\pi}{6}}$$
$$= \frac{1}{5}\left(\ln\left(\frac{2\left(-2-\sqrt{3}\right)-1}{-2-\sqrt{3}+2}\right) - \ln\left(\frac{2-1}{1+2}\right)\right)$$
$$= \frac{1}{5}\ln\left(\frac{5+2\sqrt{3}}{\sqrt{3}}\times 3\right)$$
$$= \frac{1}{5}\ln\left(6+5\sqrt{3}\right)$$

Therefore $a = 6$ and $b = 5$

28 $(x_0, y_0) = (1, 2)$

$h = \frac{1.5 - 1}{2} = 0.25$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 1^3 - 2^3 = -7$

$y_1 = y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 = 2 + 0.25 \times -7$

$y_1 = 0.25$

$(x_1, y_1) = (1.25, 0.25)$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 = 1.25^3 - 0.25^3 = 1.9375$

$y_2 = y_1 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1$

$y_2 = 0.25 + 0.25 \times 1.9375$

$y_2 = 0.734375$

Therefore $f(1.5) \approx 0.734$ (3 d.p.)

29 a $(x_0, y_0) = (\ln 2, 1)$

$y_1 = 1.6$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 2e^{\ln 2} - 1^2 = 3$

$\frac{y_1 - y_0}{h} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$\frac{1.6 - 1}{h} = 3$

$h = \frac{0.6}{3} = 0.2$

29 b $(x_1, y_1) = (\ln 2 + 0.2, 1.6)$

$\left(\frac{dy}{dx}\right)_1 = 2e^{(\ln 2 + 0.2)} - 1.6^2$

$= 2.32561\ldots$

$y_2 = y_1 + h\left(\frac{dy}{dx}\right)_1$

$y_2 = 1.6 + 0.2 \times 2.32561\ldots$

$y_2 = 2.0651\ldots$

So $y_2 \approx 2.065$ (3 d.p.)

$(x_2, y_2) = (\ln 2 + 0.4, 2.0651\ldots)$

$\left(\frac{dy}{dx}\right)_2 = 2e^{(\ln 2 + 0.4)} - (2.0651\ldots)^2$

$= 1.70257\ldots$

$y_3 = y_2 + h\left(\frac{dy}{dx}\right)_2$

$y_3 = 2.0651\ldots + 0.2 \times 1.70257\ldots$

$y_3 = 2.40564\ldots$

So $y_3 \approx 2.406$ (3 d.p.)

30 $(t_0, v_0) = (3, 8)$

$h = \frac{5-3}{2} = 1$

$\left(\frac{dv}{dt}\right)_0 = \frac{2 \times 8 - 3 \times 3}{8^2 \times 3 - 3^3} = 0.04242\ldots$

$v_1 = v_0 + h\left(\frac{dv}{dt}\right)_0$

$v_1 = 8 + 1 \times 0.04242\ldots$

$v_1 = 8.04242\ldots$

$(t_1, v_1) = (4, 8.04242\ldots)$

$\left(\frac{dv}{dt}\right)_1 = \frac{2 \times 8.04242\ldots - 3 \times 4}{(8.04242\ldots)^2 \times 4 - 4^3} = 0.020978\ldots$

$v_2 = v_1 + h\left(\frac{dv}{dt}\right)_1$

$v_2 = 8.04242\ldots + 1 \times 0.020978\ldots$

$v_2 = 8.06340\ldots$

So $v_2 \approx 8.063$ (3 d.p.)

Therefore, the value of the asset is £8063 (rounded off to the nearest pound) five days after it is purchased.

31 $h = 0.1$

$(x_0, y_0) = (1, 2)$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 2\ln 1 - 2 = -2$

$y_1 = y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 = 2 + 0.1 \times -2$

$y_1 = 1.8$

$(x_1, y_1) = (1.1, 1.8)$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1 = 2\ln 1.1 - 1.8$

$= -1.60938\ldots$

$y_2 = y_0 + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_1$

$y_2 = 2 + 2 \times 0.1 \times -1.609\,38\ldots$

$y_2 = 1.67812\ldots$

$(x_2, y_2) = (1.2, 1.678\,12\ldots)$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_2 = 2\ln 1.2 - 1.678\,12\ldots$

$= -1.313\,48\ldots$

$y_3 = y_1 + 2h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_2$

$y_3 = 1.8 + 2 \times 0.1 \times -1.313\,48\ldots$

$y_3 = 1.537\,30\ldots$

Therefore, at $x = 1.3$, $y \approx 1.537$ (3 d.p.)

32 a $h = 0.2$

$(x_0, y_0) = (1, 1)$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = \cos\left(1^2 \times 1\right)$

$= 0.540\,30\ldots$

$y_1 = y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 = 1 + 0.2\cos(1)$

$y_1 = 1 + 0.2 \times 0.540\,30\ldots$

$y_1 = 1.108\,06\ldots \approx 1.108\,(3\,\text{d.p.})$

32 b $(x_1, y_1) = (1.2, 1.108\,06\ldots)$

$\left(\frac{dy}{dx}\right)_1 = \cos(1.2^2 \times 1.108\,06\ldots)$

$= -0.02481\ldots$

$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$y_2 = 1 + 2 \times 0.2 \times -0.024\,81\ldots$

$y_2 = 0.990\,08\ldots$

$(x_2, y_2) = (1.4, 0.990\,08\ldots)$

$\left(\frac{dy}{dx}\right)_2 = \cos(1.4^2 \times 0.990\,08\ldots)$

$= -0.36139\ldots$

$y_3 = y_1 + 2h\left(\frac{dy}{dx}\right)_2$

$y_3 = 1.108\,06\ldots + 2 \times 0.2 \times -0.361\,39\ldots$

$y_3 = 0.963\,51$

$(x_3, y_3) = (1.6, 0.963\,51\ldots)$

Therefore, at $x = 1.6$, $y \approx 0.964$ (3 d.p.)

33 $(t_0, P_0) = (0, 1000)$

$h = 1 \qquad \left(\frac{dP}{dt}\right)_0 = 1000 - 0.000\,02 \times 1000^2 - 0.5\cos(0.8 \times 0)$

$= 979.5$

$P_1 = P_0 + h\left(\frac{dP}{dx}\right)_0$

$P_1 = 1000 + 1 \times 979.5$

$P_1 = 1979.5$

$(t_1, P_1) = (1, 1979.5)$

$\left(\frac{dP}{dt}\right)_1 = 1979.5 - 0.000\,02 \times 1979.5^2 - 0.5\cos(0.8 \times 1) = 1900.78...$

$P_2 = P_0 + 2h\left(\frac{dP}{dt}\right)_1$

$P_2 = 1000 + 2 \times 1 \times 1900.78\ldots$

$P_2 = 4801.57\ldots$

$(t_2, P_2) = (2, 4801.57\ldots)$

$\left(\frac{dP}{dt}\right)_2 = 4801.57\ldots - 0.000\,02 \times (4801.57\ldots)^2 - 0.5\cos(0.8 \times 2) = 4340.48$

$P_3 = P_1 + 2h\left(\frac{dP}{dt}\right)_2$

$P_3 = 1979.5 + 2 \times 1 \times 4340.48\ldots$

$P_3 = 10660.46$

Therefore, at $t = 3$ days, $P \approx 10660$ bacteria.

34 a $(x_0, y_0) = (0,1)$

$\left(\frac{dy}{dx}\right)_0 = -2$

$h = 0.1$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.1 \times -2$

$y_1 = 0.8$

b $(x_1, y_1) = (0.1, 0.8)$

$\left(\frac{d^2y}{dx^2}\right)_1 = 0.1^4 + \frac{1}{0.8}$

$= 1.2501$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{d^2y}{dx^2}\right)_1$

$y_2 = 2 \times 0.8 - 1 + 0.1^2 \times 1.2501$

$y_2 = 0.612501$

$(x_2, y_2) = (0.2, 0.612501)$

$\left(\frac{d^2y}{dx^2}\right)_2 = 0.2^4 + \frac{1}{0.612\,5010}$

$= 1.63425\ldots$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{d^2y}{dx^2}\right)_2$

$y_3 = 2 \times 0.612501 - 0.8 + 0.1^2 \times 1.63425\ldots$

$y_3 = 0.44134\ldots$

$(x_3, y_3) = (0.3, 0.44134...)$

Therefore, at $x = 0.3$, $y \approx 0.4413$ (4 d.p.)

35 $(x_0, y_0) = (1, 2)$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 0.5$

$h = 0.2$

$y_1 = y_0 + h\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 = 2 + 0.2 \times 0.5$

$y_1 = 2.1$

$(x_1, y_1) = (1.2, 2.1)$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1 = 2 - \sin 1.2 + 2\cos 2.1$

$= 0.058\,27\ldots$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1$

$y_2 = 2 \times 2.1 - 2 + 0.2^2 \times 0.058\,27\ldots$

$y_2 = 2.202\,33\ldots \approx 2.202$ (3 d.p.)

$(x_2, y_2) = (1.4, 2.202\,33\ldots)$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_2 = 2 - \sin 1.4 + 2\cos 2.202\,33\ldots$

$= -1.166\,22\ldots$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_2$

$y_3 = 2 \times 20\,233\ldots - 2.1 + 0.2^2 \times -1.166\,22\ldots$

$y_3 = 2.298\,01\ldots\ldots$

$(x_3, y_3) = (1.6, 2.298\,01\ldots)$

$y_3 \approx 2.298$ (3 d.p.)

36 $(x_0, y_0) = (2, 2)$

$h = 0.2$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 3$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 2\times2\times2+3^2=17$

$\frac{y_1 - y_{-1}}{2h} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 - y_{-1} = 2\times0.2\times3$

$y_1 - y_{-1} = 1.2$

$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0$

$y_1 + y_{-1} = 2\times2+0.2^2\times17$

$y_1 + y_{-1} = 4.68$

Adding the two equations gives

$2y_1 = 1.2+4.68 = 5.88$

$y_1 = 2.94$

37 a $y = f(x) = \sqrt[3]{\sin x - \tan x}$

$h = \frac{3-2}{4} = 0.25$

x_i	y_i
2	1.45721…
2.25	1.26342…
2.5	1.10398…
2.75	0.92622…
3	0.65706…

By Simpson's rule,

$\int_2^3 \sqrt[3]{\sin x - \tan x}\,dx$

$\approx \frac{1}{3}\times0.25\begin{pmatrix}1.45721\ldots+4(1.26342\ldots\\+0.92622\ldots)+2(1.10398\ldots)\\+0.65706\ldots\end{pmatrix}$

≈ 1.09 (2 d.p.)

b Increasing the number of intervals would give a better estimate for the approximation.

38 a $\int_1^2 x\cosh x\,dx$

$h = \frac{2-1}{2} = 0.5$

x_i	y_i
1	1.54308…
1.5	3.52861…
2	7.52439…

By Simpson's rule,

$\int_1^2 x\cosh x\,dx \approx \frac{1}{3}\times 0.5(1.54308\ldots + 4\times 3.52861\ldots + 7.52439\ldots)$

≈ 3.8637 (4 d.p.)

b Integrating by parts with $u = x$ and $v = \cosh x$:

$\int_1^2 x\cosh x\,dx = [x\sinh x]_1^2 - \int_1^2 \sinh x\,dx$

$= [x\sinh x - \cosh x]_1^2$

$= \left[x\left(\frac{e^x - e^{-x}}{2}\right) - \left(\frac{e^x + e^{-x}}{2}\right)\right]_1^2 = 2\left(\frac{e^2 - e^{-2}}{2}\right) - \left(\frac{e^2 + e^{-2}}{2}\right) - 1\left(\frac{e^1 - e^{-1}}{2}\right) + \left(\frac{e^1 + e^{-1}}{2}\right)$

$= \frac{1}{2}e^2 - \frac{3}{2}e^{-2} + e^{-1}$

c Percentage error

$= \frac{\left|3.8637 - \left(\frac{1}{2}e^2 - \frac{3}{2}e^{-2} + e^{-1}\right)\right|}{\left(\frac{1}{2}e^2 - \frac{3}{2}e^{-2} + e^{-1}\right)} \times 100$

$\approx 0.11\%$ (2 d.p.)

39 a $y = \frac{1}{2}u - \frac{1}{2}x$

Differentiate throughout with respect to x

$$\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - \frac{1}{2}$$

$$\frac{dy}{dx} = x + 2y$$

transforms to

$y = \frac{1}{2}(u - x) \Rightarrow 2y = u - x$

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{2} = x + u - x = u$$

$$\frac{du}{dx} - 1 = 2u$$

$$\frac{du}{dx} = 2u + 1$$

This is a separable equation. You learnt how to solve separable equations in C4

$$\int \frac{1}{2u+1}\,du = \int 1\,dx$$

Separating the variables.

$$\frac{1}{2}\ln(2u+1) = x + A$$

$$\ln(2u+1) = 2x + B$$

Twice one arbitrary constant A is another arbitrary constant, $B = 2A$

$$e^{\ln(2u+1)} = e^{2x+B} = e^{B}e^{2x} = Ce^{2x}$$

$$2u + 1 = 4y + 2x + 1 = Ce^{2x}$$

e to an arbitrary constant is another arbitrary constant.

Here $C = e^{B}$

$$y = \frac{Ce^{2x} - 2x - 1}{4}$$

This is the general solution of the original differential equation.

b $y = 2$ at $x = 0$

$$2 = \frac{C-1}{4} \Rightarrow 8 = C - 1 \Rightarrow C = 9$$

$$y = \frac{9e^{2x} - 2x - 1}{4}$$

This is the particular solution of the original differential equation for which $y = 2$ at $x = 0$

40 a $y = vx$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Differentiating vx as a product,
$\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x)$
$= x\frac{dv}{dx} + v$, as $\frac{d}{dx}(x) = 1$

Substituting $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into

equation **(1)** in the question

$$x\frac{dv}{dx} + v = \frac{(4x+vx)(x+vx)}{x^2}$$
$$= \frac{x^2(4+v)(1+v)}{x^2} = (4+v)(1+v) = 4+5v+v^2$$

$$x\frac{dv}{dx} = 4+4v+v^2 = (2+v)^2, \text{ as required.}$$

b $$\int \frac{1}{(2+v)^2}\,dv = \int \frac{1}{x}\,dx$$

This is a separable equation and the first step in its solution is to separate the variables, by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side of the equation.

$$-\frac{1}{2+v} = \ln x + c$$

$\int (2+v)^{-2}\,dv = \frac{(2+v)^{-1}}{-1} = -\frac{1}{2+v}$

$$2+v = -\frac{1}{\ln x + c}$$
$$v = -2 - \frac{1}{\ln x + c}$$

c $y = vx \Rightarrow v = \frac{y}{x}$

Substituting $v = \frac{y}{x}$ into the answer to part **b**

$$\frac{y}{x} = -2 - \frac{1}{\ln x + c}$$
$$y = -2x - \frac{x}{\ln x + c}\text{, as required}$$

Multiply throughout by x to obtain the printed answer.

41 a $y = vx$

$$\frac{dy}{dx} = x\frac{dx}{dx} + v$$

Differentiating vx as a product,
$\frac{d}{dx}(vx)$
$= \frac{dv}{dx}x + v\frac{d}{dx}(x)$
$= x\frac{dv}{dx} + v$, as $\frac{d}{dx}(x) = 1$

Substitute $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into

equation **(1)** in the question

$$x\frac{dv}{dx} + v = \frac{3x - 4vx}{4x + 3vx} = \frac{\not{x}(3 - 4v)}{\not{x}(4 + 3v)}$$

$$x\frac{dv}{dx} = \frac{3 - 4v}{4 + 3v} - v = \frac{3 - 4v - 4v - 3v^2}{4 + 3v} = \frac{3 - 8v - 3v^2}{4 + 3v}$$

$$x\frac{dv}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4}, \text{ as required}$$

This is a separable equation and in part **b** you solve it by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side.

b $$\int \frac{3v + 4}{3v^2 + 8v - 3}\,dv = \frac{1}{2}\int \frac{6v + 8}{3v^2 + 8v - 3}\,dv = -\int \frac{1}{x}\,dx$$

$\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)$ is a standard formula you should know. As $6v + 8$ is the derivative of $3v^2 + 8v - 3$,

$\int \frac{6v + 8}{3v^2 + 8v - 3}\,dv = \ln(3v^2 + 8v - 3)$

$$\frac{1}{2}\ln(3v^2 + 8v - 3) = -\ln x + A$$

$$\ln(3v^2 + 8v - 3) = -2\ln x + B$$

$$= \ln\frac{1}{x^2} + \ln C = \ln\frac{C}{x^2}$$

An arbitrary constant B can be written as the logarithm of another arbitrary constant $\ln C$.

Hence

$$3v^2 + 8v - 3 = \frac{C}{x^2}$$

c $y = xv \Rightarrow v = \frac{y}{x}$

Substituting into the answer to part **b**

$$\frac{3y^2}{x^2} + \frac{8y}{x} - 3 = \frac{C}{x^2}$$

$$3y^2 + 8yx - 3x^2 = C$$

Multiply each term in the equation by x^2

$y = 7$ at $x = 1$

$3 \times 49 + 56 - 3 = C \Rightarrow C = 200$

Factorising the left hand side of the equation

$(3y - x)(y + 3x) = 200$, as required.

42 a $\mu = y^{-2}$

$$\frac{d\mu}{dx} = -2 \times y^{-3} \times \frac{dy}{dx}$$

Differentiate both sides implicitly with respect to x

Hence

$$\frac{dy}{dx} = -\frac{y^3}{2}\frac{d\mu}{dx}$$

You transform this equation, making $\frac{dy}{dx}$ the subject of the formula as you need to substitute for $\frac{dy}{dx}$ in **(1)**

Substituting in equation **(1)** in the question

$$-\frac{y^3}{2}\frac{d\mu}{dx} - 2xy = xe^{-x^2}y^3$$

Divide by y^3

$$-\frac{1}{2}\frac{d\mu}{dx} + \frac{2x}{y^2} = xe^{-x^2}$$

As $\mu = \frac{1}{y^2}$

$$-\frac{1}{2}\frac{d\mu}{dx} + 2x\mu = xe^{-x^2}$$

Multiply by (–2)

$$\frac{d\mu}{dx} - 4x\mu = -2xe^{-x^2}$$, as required

42 b The integrating factor of **(2)** is

$$e^{\int -4x\,dx} = e^{-2x^2}$$

Multiplying **(2)** throughout by e^{-2x^2}

$$e^{-2x^2}\frac{d\mu}{dx} - 4x\mu e^{-2x^2} = -2xe^{-x^2} \times e^{-2x^2} = -2xe^{-3x^2}$$

$$\frac{d}{dx}(\mu e^{-2x^2}) = -2xe^{-3x^2}$$

$$\mu e^{-2x^2} = -2\int xe^{-3x^2}dx = \frac{1}{3}e^{-3x^2} + C$$

This integration can be carried out by inspection. As $\frac{d}{dx}(e^{-3x^2}) = -6xe^{-3x^2}$, then $\int xe^{-3x^2}dx = -\frac{1}{6}e^{-3x^2}$

Multiplying throughout by e^{2x^2}

$$\mu = \frac{1}{3}e^{-x^2} + Ce^{2x^2}$$

c As $\mu = \frac{1}{y^2}$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + Ce^{2x^2}$$

$y = 1$ at $x = 0$

$$1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + \frac{2}{3}e^{2x^2}$$

As no form of the answer has been specified in the question, this is an acceptable answer for the particular solution of **(1)**

43 a $y = xv$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 2\frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2v}{\mathrm{d}x^2}$$

Use the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(xv) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \times v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \times v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

Substituting for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ into **(1)**

$$x^2\left(x\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\frac{\mathrm{d}v}{\mathrm{d}x}\right) - 2x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (2 + 9x^2)vx = x^5$$

$$x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \cancel{2x^2\frac{\mathrm{d}v}{\mathrm{d}x}} - \cancel{2xv} - \cancel{2x^2\frac{\mathrm{d}v}{\mathrm{d}x}} + \cancel{2xv} + 9x^3v = x^5$$

$$x^3\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 9x^3v = x^5$$

Divide by x^3

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 9v = x^2\text{, as required}$$

43 b The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m^2 = -9$$
$$m = \pm\, 3\text{i}$$

The complementary function is given by

$$v = A\cos 3x + B\sin 3x$$

For a particular integral, try $v = px^2 + qx + r$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is a quadratic x^2, so you try a general quadratic $px^2 + qx + r$

$$\frac{\text{d}v}{\text{d}x} = 2px + q,\ \frac{\text{d}^2 v}{\text{d}x^2} = 2p$$

Substituting into **(2)**

$$2p + 9qx^2 + 9qx + 9r = x^2$$

Equating coefficients of x^2

$$9p = 1 \Rightarrow q = \frac{1}{9}$$

Equating coefficient of x

$$9q = 0 \Rightarrow q = 0$$

Equating constant coefficients

As $p = \frac{1}{9}$

$$2p + 9r = 0 \Rightarrow 9r = -2p = -\frac{2}{9} \Rightarrow r = -\frac{2}{81}$$

A particular integral is $\frac{1}{9}x^2 - \frac{2}{8!}$

A general solution of **(2)** is

$$v = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$$

$y = vx \Rightarrow v = \frac{y}{x}$

c $$\frac{y}{x} = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$$

The question does not ask for a particular form of the answer in part **c**, so this would be an acceleration answer.

$$y = Ax\cos 3x + Bx\sin 3x + \frac{1}{9}x^3 - \frac{2}{81}x$$

44 a $x = t^{\frac{1}{2}} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{2}t^{-\frac{1}{2}} = \dfrac{1}{2t^{\frac{1}{2}}}$

$$\frac{dt}{dx} = \frac{1}{\frac{1}{2t^{\frac{1}{2}}}} = 2t^{\frac{1}{2}}$$

Use $\dfrac{dt}{dx} = \dfrac{1}{\frac{dx}{dt}}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times 2t^{\frac{1}{2}} = 2t^{\frac{1}{2}}\frac{dy}{dt}$$

You obtain an expression for $\dfrac{dy}{dx}$ using the chain rule.

b Substituting $x = t^{\frac{1}{2}}$, the result of part **a** and the

given $\dfrac{d^2y}{dx^2} = 4t\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt}$ into **(1)**

$$\left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right) = 2t^{\frac{1}{2}} = 6t^{\frac{1}{2}} \times 2t^{\frac{1}{2}} - \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = 12t - 2$$

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right)2t^{\frac{1}{2}}\frac{dy}{dt} - 16ty = 4t\,e^{2t}$$

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 12t\frac{dy}{dt} - 2\frac{dy}{dt} - 16ty = 4t\,e^{2t}$$

$$4t\frac{d^2y}{dt^2} + 12t\frac{dy}{dt} - 16ty = 4t\,e^{2t}$$

Divide throughout by $4t$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}, \text{ as required}$$

44 c The auxiliary equation is

$$m^2+3m-4=(m-1)(m+4)=0$$
$$m=1,\ -4$$

The complementary function is

$$y=Ae^{t}+Be^{-4t}$$

For a particular integral try, $y=k\,e^{2t}$

If the right hand side of the equation is $e^{\alpha t}$, you can try $k\,e^{\alpha t}$ as a particular integral. This will work unless α is a solution of the auxiliary equation.

$$\frac{dy}{dt}=2k\,e^{2t},\ \frac{d^2y}{dt^2}=4k\,e^{2t}$$

Substituting into $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-4y=e^{2t}$

$$\cancel{4k\,e^{2t}}+6k\,e^{2t}\cancel{-4k\,e^{2t}}=e^{2t}$$
$$6k=1\Rightarrow k=\frac{1}{6}$$

As e^{2t} cannot be zero, you can divide throughout by e^{2t}

A particular integral is $\frac{1}{6}e^{2t}$

The general solution of the differential equation in y and t is

$$y=Ae^{t}+Be^{-4t}+\frac{1}{6}e^{2t}$$

$$x=t^{\frac{1}{2}}\Rightarrow t=x^2$$

The general solution of **(1)** is

$$y=Ae^{x^2}+Be^{-4x^2}+\frac{1}{6}e^{2x^2}$$

45 a $x=\ln t\Rightarrow\frac{dx}{dt}=\frac{1}{t}\Rightarrow\frac{dt}{dx}=t$

$$\frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{dy}{dt}\times t$$
$$\frac{dy}{dx}=t\frac{dy}{dt}$$

45 b $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx} \times \frac{d}{dt}\left(\frac{dy}{dx}\right)$

$= t\frac{d}{dt}\left(t\frac{dy}{dt}\right)$

$= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right)$

$= t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$, as required

It is a common error to proceed from $\frac{dy}{dx} = t\frac{dy}{dt}$ to $\frac{d^2y}{dx^2} = \frac{dy}{dt} + t\frac{d^2y}{dt^2}$

This is incorrect because the left hand side has been differentiated with respect to x and the right hand side with respect to t. The version of the chain rule given here must be used.

c Substituting $x = \ln t, \frac{dy}{dx} = t\frac{dy}{dt}$ and

$\frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$ into **(1)**

$e^{2\ln t} = e^{\ln t^2} = t^2$, using the log role $n \ln a = \ln a^n$ and $e^{\ln f(t)} = f(t)$

$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (1-6t)t\frac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$

$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - t\frac{dy}{dt} + 6t^2\frac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$

After cancelling, divide throughout by t^2

$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5\sin 2t$, as required

45 d The auxiliary equation of **(1)** is

$$m^2 + 6m + 10 = 0$$
$$m^2 + 6m + 9 = -1$$
$$(m+3)^2 = -1$$
$$m + 3 = \pm \mathrm{i}$$
$$m = -3 \pm \mathrm{i}$$

The complementary function is given by

$$y = \mathrm{e}^{-3t}(A\cos t + B\sin t)$$

For a particular integral try $y = p\sin 2t + q\cos 2t$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2p\cos 2t - 2p\sin 2t$$
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -4p\sin 2t - 4q\cos 2t$$

If the right hand side of the second order differential equation is a k sin nt or k cos nt function, then you should try a particular integral of the form $p\cos nt + q\sin nt$

Substituting into **(2)**

$$-4p\sin 2t - 4q\cos 2t + 12p\cos 2t - 12q\sin 2t + 10p\sin 2t + 10q\cos 2t = 5\sin 2t$$
$$(-4q - 12q + 10p)\sin 2t + (-4q + 12q + 10q)\cos 2t = 5\sin 2t$$
$$(6p - 12q)\sin 2t + (12p + 6q)\cos 2t = 5\sin 2t$$

Equating the coefficients of sin $2t$

$$6p - 12q = 5 \qquad \textbf{(3)}$$
$$12p + 6q = 0 \qquad \textbf{(4)}$$

You can solve the simultaneous equations by any appropriate method.

From **(4)** $p = -\frac{6}{12}q = -\frac{1}{2}q$

Substitute into **(3)**

$$-3q - 12q = -15q = 5 \Rightarrow q = -\frac{1}{3}$$

Hence $p = -\frac{1}{2}q = -\frac{1}{2} \times -\frac{1}{3} = \frac{1}{6}$

The general solution of **(2)** is

$$y = \mathrm{e}^{-3t}(A\cos t + B\sin t) + \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t$$

$$x = \ln t \Rightarrow t = \mathrm{e}^x$$

45 d The general solution of **(1)** is

$$y = e^{-3e^x}(A\cos(e^x) + B\sin(e^x)) + \frac{1}{6}\sin(2e^x) - \frac{1}{3}\cos(2e^x)$$

46 a $y = x^{-2}$

Differentiating implicitly with respect to t.

$$\frac{dy}{dt} = -2x^{-3}\frac{dx}{dt}$$

Use $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

Differentiating again implicitly with respect to t.

$$\frac{d^2y}{dt^2} = 6x^{-4}\left(\frac{dx}{dt}\right)^2 - 2x^{-3}\frac{d^2x}{dt^2} \quad \textbf{(2)}$$

This expression is closely related to the left hand side of the original differential equation in the question. This suggests to you that if you divide the original equation by $-x^4$, then the left hand side can just be replaced by $\frac{d^2x}{dt^2}$

Dividing the differential equation given in the question by $-x^4$, it becomes

$$-2x^{-3}\frac{d^2x}{dt^2} + 6x^{-4}\left(\frac{dx}{dt}\right)^2 = -x^{-2} + 3$$

Using equation **(2)** and $y = x^{-2}$

$$\frac{d^2y}{dt^2} = -y + 3$$

$$\frac{d^2y}{dt^2} + y = 3, \text{ as required}$$

b The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The complementary function is given by

$$y = A\cos t + B\sin t$$

By inspection, a particular integral of **(1)** is 3

As $\frac{d^2}{dt^2}(3) = 0$, $y = 3$ satisfies $\frac{d^2y}{dt^2} + y = 3$, by inspection and you need not write down any working.

The general solution of **(2)** is

$$y = A\cos t + B\sin t + 3$$

46 c The general solution of the differential equation in x and t is

$$\frac{1}{x^2} = A\cos t + B\sin t + 3 \qquad \textbf{(3)}$$

When $t = 0,\ x = \frac{1}{2}$

$$4 = A + 3 \Rightarrow A = 1$$

Differentiating **(3)** implicitly with respect to t

$$-\frac{2}{x^3}\frac{\mathrm{d}x}{\mathrm{d}t} = -A\sin t + B\cos t$$

Use the chain rule
$\frac{\mathrm{d}}{\mathrm{d}t}(x^{-2}) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{-2}) \times \frac{\mathrm{d}x}{\mathrm{d}t} = -2x^{-3}\frac{\mathrm{d}x}{\mathrm{d}t}$

When $t = 0,\ x = \frac{1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

$$0 = B$$

The particular solution is

$$\frac{1}{x^2} = \cos t + 3$$

As $x > 0,\ t > 0$

$$x = \frac{1}{\sqrt{(\cos t + 3)}}$$

As x and t are both positive, the negative square root need not be considered.

d The maximum value of x is

$$x = \frac{1}{\sqrt{(-1+3)}} = \frac{1}{\sqrt{2}}$$

The maximum value of this fraction is when the denominator has its least value. The smallest possible value of $\cos t$ is -1

So you can write down the maximum value without using calculus.

Challenge

1 $t = \tan\left(\frac{x}{2}\right)$ and $s = \tan\left(\frac{y}{2}\right)$

$$\frac{\tan x + \tan y}{\cot x + \cot y} \equiv \frac{\frac{2t}{1-t^2} + \frac{2s}{1-s^2}}{\frac{1-t^2}{2t} + \frac{1-s^2}{2s}}$$

$$\equiv \frac{2t\left(1-s^2\right) + 2s\left(1-t^2\right)}{\left(1-t^2\right)\left(1-s^2\right)} \div \frac{\left(1-t^2\right)\left(1-s^2\right)}{4ts}$$

$$\equiv \frac{4ts}{\left(1-t^2\right)\left(1-s^2\right)}$$

$$\equiv \left(\frac{2t}{1-t^2}\right)\left(\frac{2s}{1-s^2}\right) = \tan x \tan y$$

2 Let $u = e^x \cosh x$ and $v = x^3$

$$\frac{dv}{dx} = 3x^2,\ \frac{d^2v}{dx^2} = 6x,\ \frac{d^3v}{dx^3} = 6$$

The subsequent derivatives of v are all zero.

$$u = e^x \cosh x = e^x\left(\frac{e^x + e^{-x}}{2}\right) = \frac{1}{2}\left(e^{2x} + 1\right)$$

$$\frac{d^nu}{dx^n} = 2^{n-1}e^{2x}$$

Using Leibnitz's theorem,

$$\frac{d^ny}{dx^n} = u\frac{d^nv}{dx^n} + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \ldots +$$

$$\frac{n(n-1)(n-2)}{6}\frac{d^{n-3}u}{dx^{n-3}}\frac{d^3v}{dx^3} + \frac{n(n-1)}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}$$

$$+n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \frac{d^nu}{dx^n}v$$

$$= n(n-1)(n-2)2^{n-4}e^{2x} + n(n-1)2^{n-3}e^{2x}3x$$

$$+n2^{n-2}e^{2x}3x^2 + 2^{n-1}e^{2x}x^3$$

$$= 2^{n-4}e^{2x}\left(8x^3 + 12nx^2 + 6n(n-1)x + n(n-1)(n-2)\right)$$

Challenge

3 a $f''(x) = (f'(x))^3$, $u = f'(x)$

$\Rightarrow u' = u^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = u^3 \Rightarrow \frac{\mathrm{d}u}{u^3} = \mathrm{d}x$

$\int \frac{\mathrm{d}u}{u^3} = \int \mathrm{d}x$

$\frac{1}{2u^2} = -x + B$

$u^2 = \frac{1}{-2x + B}$

$u = \frac{1}{\sqrt{B - 2x}} \Rightarrow f'(x) = \frac{1}{\sqrt{B - 2x}}$

Integrating both sides with respect to x,

$f(x) = \int \frac{1}{\sqrt{B - 2x}} \mathrm{d}x = A - \sqrt{B - 2x}$

b $f(0) = 0 \Rightarrow A - \sqrt{B} = 0$

$f(1) = 1 \Rightarrow A - \sqrt{B - 2} = 1$

Subtracting the two equations gives

$\sqrt{B} - \sqrt{B - 2} = 1$

$\sqrt{B} = 1 + \sqrt{B - 2}$

$B = \left(1 + \sqrt{B - 2}\right)^2 = 1 + B - 2 + 2\sqrt{B - 2}$

$\sqrt{B - 2} = \frac{1}{2} \Rightarrow B - 2 = \frac{1}{4} \Rightarrow B = \frac{9}{4}$

$A - \sqrt{B} = 0 \Rightarrow A = \sqrt{\frac{9}{4}} = \frac{3}{2}$