Reducible differential equations 9 Mixed Exercise

1 a Given that $z = y^{-1}$, then $y = z^{-1}$ so $\frac{dy}{dx} = -z^{-2}\frac{dz}{dx}$

The equation $x\frac{dy}{dx} + y = y^2 \text{In} x$ becomes

$-xz^{-2}\frac{dz}{dx} + z^{-2}\text{In} x$

Dividing through by $-xz^{-2}$ gives $\frac{dz}{dx} - \frac{z}{x} = -\frac{\text{In} x}{x}$

b Integrating factor is $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

So the differential equation becomes

$\frac{d}{dx}\left(\frac{z}{x}\right) = -\frac{\ln x}{x^2}$

$\Rightarrow \frac{z}{x} = \int -\frac{\ln x}{x^2}dx$

Using integration by parts

$u = -\ln x, \frac{dv}{dx} = \frac{1}{x^2} \Rightarrow \frac{du}{dx} = -\frac{1}{x}, v = -\frac{1}{x}$

So $\int -\frac{\ln x}{x^2}dx = \frac{\ln x}{x} - \int \frac{1}{x^2}dx$

$\Rightarrow \frac{z}{x} = \frac{\ln x}{x} + \frac{1}{x} + c$

$\Rightarrow z = \ln x + cx + 1$

$\Rightarrow y = \frac{1}{1 + cx + \ln x}$

2 a Given that $= y^2, y = z^{\frac{1}{2}}$ and $\frac{dy}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx'}$

the differential equation becomes

$\cos x\, z^{-\frac{1}{2}}\frac{dz}{dx} - z^{\frac{1}{2}}\sin x + z^{-\frac{1}{2}} = 0$

Divide through by $z^{-\frac{1}{2}}$: $\cos x\frac{dz}{dx} - z\sin x = -1$

b Differential equation is $\frac{dz}{dx} - \frac{\sin x}{\cos x}z = -\sec x$

Integrating factor is $e^{-\int \frac{\sin x}{\cos x}dx} = e^{\ln \cos x} = \cos x$

So $\frac{d}{dx}(z\cos x) = -1$

$\Rightarrow z\cos x = -x + c$

$\Rightarrow z = -x\sec x + c\sec x$

c $y^2 = c\sec x - x\sec x$

3 a Given that $z=\dfrac{y}{x}, y=zx$ so $\dfrac{\mathrm{d}y}{\mathrm{d}x}=z+x\dfrac{\mathrm{d}z}{\mathrm{d}x}$

The equation $\left(x^2-y^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}-xy=0$ becomes

$$\left(x^2-z^2x^2\right)\left(z+x\frac{\mathrm{d}z}{\mathrm{d}x}\right)-xzx=0$$

$$\Rightarrow\left(1-z^2\right)z+\left(1-z^2\right)x\frac{\mathrm{d}z}{\mathrm{d}x}-z=0$$

$$\Rightarrow x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z}{1-z^2}-z$$

$$\Rightarrow x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z^3}{1-z^2}$$

b Separating the variables the equation becomes

$$\int\frac{1-z^2}{z^3}\mathrm{d}z=\int\frac{1}{x}\mathrm{d}x$$

$$\Rightarrow-\frac{1}{2}z^{-2}-\ln z=\ln x+c$$

$$\Rightarrow-\frac{1}{2}\frac{x^2}{y^2}-\ln\frac{y}{x}=\ln x+c$$

$$\Rightarrow-\frac{1}{2}\frac{x^2}{y^2}-\ln y+\ln x=\ln x+c$$

$$\Rightarrow-\frac{1}{2}\frac{x^2}{y^2}-\ln y=c$$

$$\Rightarrow 2y^2(\ln y+c)+x^2=0$$

4 a $z=\dfrac{y}{x}\Rightarrow y=xz$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=z+x\dfrac{\mathrm{d}z}{\mathrm{d}x}$

So $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y(x+y)}{x(y-x)}$ becomes $z+x\dfrac{dz}{dx}=\dfrac{xz(x+xz)}{x(xz-x)}$

$$\Rightarrow z+x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{z(1+z)}{z-1}$$

So $x\dfrac{\mathrm{d}z}{\mathrm{d}x}=\dfrac{z(1+z)}{z-1}-z=\dfrac{2z}{z-1}$

4 b Separating the variables,

$$\int \frac{z-1}{2z}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

$$\Rightarrow \frac{1}{2}z - \frac{1}{2}\ln z = \ln x + c$$

$$\Rightarrow \frac{1}{2}\frac{y}{x} - \frac{1}{2}\ln\frac{y}{x} = \ln x + c$$

$$\Rightarrow \frac{1}{2}\frac{y}{x} - \frac{1}{2}\ln y + \frac{1}{2}\ln x = \ln x + c$$

$$\Rightarrow \frac{y}{2x} - \frac{1}{2}\ln y = \frac{1}{2}\ln x + c$$

5 a Given that $z = \frac{y}{x}$, $y = zx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = z + x\frac{\mathrm{d}z}{\mathrm{d}x}$

The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-3xy}{y^2 - 3x^2}$ becomes

$$z + x\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{-3z}{z^2-3} - z = \frac{-z^3}{z^2-3}$$

b Separating the variables,

$$\int -\frac{z^2-3}{z^3}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

$$\Rightarrow -\ln z - \frac{3}{2}z^{-2} = \ln x + c$$

$$\Rightarrow -\ln y + \ln x - \frac{3}{2}\frac{x^2}{y^2} = \ln x + c$$

$$\Rightarrow -\ln y - \frac{3}{2}\frac{x^2}{y^2} = c$$

$\Rightarrow \ln y + \frac{3x^2}{2y^2} = c$ (since c is an arbitrary constant, we can change the sign.)

6 a Let $u = x + y$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$ and so

$\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y+1)(x+y-1)$ becomes

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = (u+1)(u-1) = u^2 - 1$$

$$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = u^2$$

6 b $\int \frac{1}{u^2} \mathrm{d}u = \int \mathrm{d}x$

$\Rightarrow -\frac{1}{u} = x + c$

$\Rightarrow u = \frac{-1}{x+c}$

$\Rightarrow x + y = \frac{-1}{x+c}$

$\Rightarrow y = \frac{-1}{x+c} - x$

7 a Given that $u = y - x - 2$, $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} - 1$

So $\frac{\mathrm{d}y}{\mathrm{d}x} = (y - x - 2)^2$ becomes $\frac{\mathrm{d}u}{\mathrm{d}x} + 1 = u^2$

$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = u^2 - 1$

b $\int \frac{\mathrm{d}u}{u^2 - 1} = x$

$\Rightarrow \frac{1}{2} \ln \frac{u-1}{u+1} = x + c$

$\Rightarrow \frac{u-1}{u+1} = \mathrm{e}^{2(x+c)}$

$\Rightarrow \frac{u-1}{u+1} = A\mathrm{e}^{2x}$ writing A as e^{2c}

so A is positive

$\Rightarrow u - 1 = A\mathrm{e}^{2x}(u+1)$

$\Rightarrow u = \frac{1 + A\mathrm{e}^{2x}}{1 - A\mathrm{e}^{2x}}$

$\Rightarrow y - x - 2 = \frac{1 + A\mathrm{e}^{2x}}{1 - A\mathrm{e}^{2x}}$

$\Rightarrow y = x + 2 + \frac{1 + A\mathrm{e}^{2x}}{1 - A\mathrm{e}^{2x}}$

8 a $v = u^{-\frac{1}{2}}, \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{2} u^{-\frac{3}{2}} \frac{\mathrm{d}u}{\mathrm{d}t}$

Equation becomes $-\frac{1}{2} u^{-\frac{3}{2}} \frac{\mathrm{d}u}{\mathrm{d}t} \times t + u^{-\frac{1}{2}} = 2t^3 u^{-\frac{3}{2}}$ which rearranges to $\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{2u}{t} = -4t^2$

8 b Using integrating factor $e^{-2\int \frac{1}{t}dt} = e^{-2\ln t} = t^{-2}$, get

$\frac{d}{dt}\left(ut^{-2}\right) = -4 \Rightarrow ut^{-2} = -4t + c$, and $u = -4t^3 + ct^2$

Then the general solution for the original equation is $v = \frac{1}{\sqrt{t^2(c-4t)}}$

Given that $v = \frac{1}{2}$ when $t = 1$, $\frac{1}{\sqrt{c-4}} = \frac{1}{2}$, so $c = 8$ and the particular solution is $v = \frac{1}{\sqrt{t^2(8-4t)}}$

9 a Let $x = e^u$, then $\frac{dx}{du} = e^u$

and $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = e^u \frac{dy}{dx} = x\frac{dy}{dx}$

$$\frac{d^2y}{du^2} = \frac{dx}{du} \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} \times \frac{dx}{du}$$

$$= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

Find $\frac{dy}{du}$ in terms of x and $\frac{dy}{dx}$, and show that $\frac{d^2y}{du^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$ then substitute into the differential equation.

$\therefore \quad x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \ln x \Rightarrow \frac{d^2y}{du^2} + 3\frac{dy}{du} + 2y = \ln x = u$ *

The auxiliary equation is

$m^2 + 3m + 2 = 0$

$\therefore \quad (m+2)(m+1) = 0$

$\Rightarrow \quad m = -1$ or -2

$\therefore$ The c.f. is $y = Ae^{-u} + Be^{-2u}$

Let the p.i. be $y = \lambda u + \mu \Rightarrow \frac{dy}{du} = \lambda$, $\frac{d^2y}{du^2} = 0$

Substitute into *

$\therefore \; 3\lambda + 2\lambda u + 2\mu = u$

Equate coefficients of u: $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

constants: $3\lambda + 2\mu = 0 \quad \therefore \quad \mu = -\frac{3}{4}$

$\therefore$ The p.i. is $y = \frac{1}{2}u - \frac{3}{4}$

The general solution is $y = Ae^{-u} + Be^{-2u} + \frac{1}{2}u - \frac{3}{4}$

But $x = e^u \rightarrow u = \ln x$

Also $e^{-u} = x^{-1} = \frac{1}{x}$ and $e^{-2u} = x^{-2} = \frac{1}{x^2}$

$\therefore$ The general solution of the original equation is $y = \frac{A}{x} + \frac{B}{x^2} + \frac{1}{2}\ln x - \frac{3}{4}$

9 b But $y = 1$ when $x = 1$

$\therefore \quad 1 = A + B - \frac{3}{4} \Rightarrow A + B = 1\frac{3}{4}$ **(1)**

$$\frac{dy}{dx} = -\frac{A}{x^2} - \frac{2B}{x^3} + \frac{1}{2x}$$

When $x = 1, \frac{dy}{dx} = 1$

$\therefore \quad 1 = -A - 2B + \frac{1}{2} \Rightarrow A + 2B = -\frac{1}{2}$ **(2)**

Solve the simultaneous equations **(1)** and **(2)** to give $B = -2\frac{1}{4}$ and $A = 4$

$\therefore$ The equation of the solution curve described is $y = \frac{4}{x} - \frac{9}{4x^2} + \frac{1}{2}\ln x - \frac{3}{4}$

10 $z = \sin x \quad \therefore \quad \frac{dz}{dx} = \cos x$ and $\frac{dy}{dx} = \frac{dy}{dz} \times \cos x$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{dy}{dz}\sin x + \cos x \frac{d^2y}{dz^2} \times \frac{dz}{dx}$$

$$= -\frac{dy}{dz}\sin x + \cos^2 x \frac{d^2y}{dz^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = \cos^2 x\, e^{\sin x} \quad \dagger$$

$$\Rightarrow \cos^2 x \frac{d^2y}{dz^2} - \sin x \frac{dy}{dz} + \tan x \cos x \frac{dy}{dz} + y\cos^2 x = \cos^2 x\, e^z$$

$$\Rightarrow \frac{d^2y}{dz^2} + y = e^z \quad *$$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm\, i$

$\therefore$ The c.f. is $y = A\cos z + B\sin z$

The p.i. is $y = \lambda e^z \Rightarrow \frac{dy}{dz} = \lambda e^z$ and $\frac{d^2y}{dz^2} = \lambda e^z$

Substitute in $*$ to give

$$2\lambda e^z = e^z \Rightarrow \lambda = \frac{1}{2}$$

$\therefore$ The general solution of $*$ is $y = A\cos z + B\sin z + \frac{1}{2}e^z$

The original equation $\dagger$ has solution

$$y = A\cos(\sin x) + B\sin(\sin x) + \frac{1}{2}e^{\sin x}$$

But $y = 1$ when $x = 0$

$$\therefore \quad 1 = A + \frac{1}{2} \Rightarrow A = \frac{1}{2}$$

$$\frac{dy}{dx} = \cos x(-A\sin(\sin x)) + \cos x(B\cos(\sin x)) + \frac{1}{2}\cos x\, e^{\sin x}$$

As $\frac{dy}{dx} = 3$ when $x = 0$

$$\therefore \quad 3 = B + \frac{1}{2} \Rightarrow B = 2\frac{1}{2}$$

$$\therefore \quad y = \frac{1}{2}\cos(\sin x) + \frac{5}{2}\sin(\sin x) + \frac{1}{2}e^{\sin x}$$

11 a $t = \mathrm{e}^u, u = \mathrm{In}\, t, \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{t}, \frac{\mathrm{d}^2u}{\mathrm{d}t^2} = -\frac{1}{t^2}$

$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}u^2} \times \frac{1}{t^2} - \frac{\mathrm{d}x}{\mathrm{d}u} \times \frac{1}{t^2}$

So equation becomes

$$t^2\left(\frac{\mathrm{d}^2x}{\mathrm{d}u^2} \times \frac{1}{t^2} - \frac{\mathrm{d}x}{\mathrm{d}u} \times \frac{1}{t^2}\right) - 2t\left(\frac{\mathrm{d}x}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}t}\right) + 2x = 4\ \mathrm{In}\left(\mathrm{e}^u\right)$$

which rearranges to $\frac{\mathrm{d}^2x}{\mathrm{d}u^2} - 3\frac{\mathrm{d}x}{\mathrm{d}u} + 2x = 4u$

b Auxiliary equation is $m^2 - 3m + 2 = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m = 2, 1$

So the general solution is

$x = Ae^{2u} + Be^u + f(u)$

where $f(u)$ is a particular integral

Putting $f(u) = au + b$ we obtain

$-3a + 2(au + b) = 4u$

$\Rightarrow a = 2, b = 3$

$\Rightarrow x = Ae^{2u} + Be^u + 2u + 3$

$= At^2 + Bt + 2\ln t + 3$

c As t gets very large, the distance of the particle from its original position becomes very large.

12 a $\frac{\mathrm{d}x}{\mathrm{d}t} = v + t\frac{\mathrm{d}v}{\mathrm{d}t}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2\frac{\mathrm{d}v}{\mathrm{d}t} + t\frac{\mathrm{d}^2v}{\mathrm{d}t^2}$

Equation becomes

$$2t^2\left(2\frac{\mathrm{d}v}{\mathrm{d}t} + t\frac{\mathrm{d}^2v}{\mathrm{d}t^2}\right) - 4t\left(v + t\frac{\mathrm{d}v}{\mathrm{d}t}\right) + \left(4 - 2t^2\right)tv = t^4$$

which rearranges to $2\frac{\mathrm{d}^2v}{\mathrm{d}t^2} - 2v = t$

12 b $\dfrac{d^2v}{dt^2} - v = \dfrac{1}{2}t$

Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

so the general solution is

$v = Ae^t + Be^{-t} + f(t)$

where $f(t)$ is a particular integral

Let $f(t) = at + b$

so equation becomes

$-(at + b) = \dfrac{1}{2}t \Rightarrow a = -\dfrac{1}{2}, b = 0$

so the general solution is

$v = Ae^t + Be^{-t} - \dfrac{1}{2}t$

$\Rightarrow x = Ate^t + Bte^{-t} - \dfrac{1}{2}t^2$

13 a $u = v^{-1} \Rightarrow \dfrac{du}{dt} = -v^{-2}\dfrac{dv}{dt}$

$\Rightarrow \dfrac{dv}{dt} = -v^2\dfrac{du}{dt}$

So the equation becomes

$-\dfrac{1000}{u^2}\dfrac{du}{dt} - \dfrac{500}{u} + \dfrac{t}{u^2} = 0$

$\Rightarrow \dfrac{du}{dt} + 0.5u = 0.001t$ as required.

b Integrating factor is $e^{0.5t}$

so the equation becomes

$\dfrac{d}{dt}\left(e^{0.5t}u\right) = 0.001te^{0.5t}$

Using integration by parts

$\int 0.001te^{0.5t}\,dt$

$= 0.001t \times 2e^{0.5t} - \int 2e^{0.5t} \times 0.001\,dt$

$= 0.002te^{0.5t} - 0.004e^{0.5t} + c$

So we obtain

$e^{0.5t}u = 0.002te^{0.5t} - 0.004e^{0.5t} + c$

$\Rightarrow u = 0.002t - 0.004 + ce^{-0.5t}$

$\Rightarrow v = \dfrac{1}{0.002t - 0.004 + ce^{-0.5t}} = \dfrac{500e^{0.5t}}{e^{0.5t}(t-2) + c}$

13 c $2 = \dfrac{1}{-0.004 + c}$

$\Rightarrow c = 0.504$

$v = \dfrac{1}{0.002t - 0.004 + 0.504e^{-0.5t}} = \dfrac{500e^{0.5t}}{e^{0.5t}(t-2) + 252}$

d $\dfrac{1}{v} = \ 0.002t - 0.004 + 0.504e^{-0.5t}$

So for large t $\approx 0.002t$

This will increase without limit

so $v \to 0$

This means that the model is not valid for large t because we would expect v to reach a terminal velocity.

Challenge

Substitute $u = \dfrac{dy}{dx}$ so equation becomes

$\dfrac{du}{dx} = u^2$

$\Rightarrow \int \dfrac{du}{u^2} = \int dx$

$\Rightarrow -\dfrac{1}{u} = x + B$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{x+B}$

$\Rightarrow y = -\ln(x+B) + A$

$= A - \ln(x+B)$ as required.