Reducible differential equations 9C

1 a If $x = ut$ then

$$\frac{dx}{dt} = \frac{du}{dt}t + u$$

So $t(ut)\left(\frac{du}{dt}t + u\right) - u^2t^2 = 3t^4$

which rearranges to $u\frac{du}{dt} = 3t$

b Solve the differential equation in u and t to get $\frac{1}{2}u^2 = \frac{3}{2}t^2 + c$, and then use $u = \frac{x}{t} = 3$ to find $c = 3$.

So $u^2 = 3t^2 + 6 \Rightarrow x^2 = 3t^4 + 6t^2 \Rightarrow x = \sqrt{3t^4 + 6t^2}$

The particular solution is $x = t\sqrt{3t^2 + 6}$

c The function increases without limit so the displacement gets very large.

2 a $\frac{dv}{dt} = \frac{dz}{dt}t + z$

So $3z^2t^3\left(\frac{dz}{dt}t + z\right) = z^3t^3 + t^3$, which

rearranges to $3z^2t\frac{dz}{dt} = 1 - 2z^3$

b Differential equation in z and t solves to give $|1 - 2z^3| = \frac{A}{t^2}$

If $v = 2$ for $t = 1$, then $z = 2$, and $A = |-15| \times 1 = 15$

Then $t^2(2z^3 - 1) = 15 \Rightarrow 2v^3 - t^3 = 15t$

The particular solution is $v = \sqrt[3]{\frac{t^3 + 15t}{2}}$

c $v = \left(\frac{t^3 + 15t}{2}\right)^{\frac{1}{3}}$

$$\Rightarrow \frac{dv}{dt} = \frac{1}{3}\left(\frac{t^3 + 15t}{2}\right)^{-\frac{2}{3}} \times \left(\frac{3t^2 + 15}{2}\right)$$

So substituting t=2 we obtain

velocity = 2.6684... ≈ 2.668

acceleration = 0.63199... ≃ 0.632

3 a $s = \frac{v}{t}, \frac{ds}{dt} = \frac{1}{t}\frac{dv}{dt} - \frac{v}{t^2}, \frac{d^2s}{dt^2}$

$$= \frac{1}{t}\frac{d^2s}{dt^2} - \frac{2}{t^2}\frac{dv}{dt} + \frac{2v}{t^3}$$

So equation becomes

$$t\left(\frac{1}{t}\frac{d^2v}{dt^2} - \frac{2}{t^2}\frac{dv}{dt} + \frac{2v}{t^3}\right) + (2 - t)\left(\frac{1}{t}\frac{dv}{dt} - \frac{v}{t^2}\right) -$$

$$(1 + 2t)\frac{v}{t} = e^{2t}$$

Rearranging terms gives

$$\frac{d^2v}{dt^2} + \left(-\frac{2}{t} + \frac{2 - t}{t}\right)\frac{dv}{dt} + \left(\frac{2v}{t^2} - \frac{(2 - t)v}{t^2} - \frac{(1 + 2t)v}{t}\right)$$

which simplifies to $\frac{d^2v}{dt^2} - \frac{dv}{dt} - 2v = e^{2t}$

b Auxiliary equation has roots 2 and −1, so the complementary function is $v = Ae^{2t} + Be^{-t}$. To find the particular integral, try $v = \lambda te^{2t}$

Then

$$\frac{dv}{dt} = \lambda e^{2t} + 2\lambda te^{2t} \text{ and } \frac{d^2v}{dt^2} = 4\lambda e^{2t} + 4\lambda te^{2t}$$

$$\text{So } \frac{d^2v}{dt^2} - \frac{dv}{dt} - 2v = 4\lambda e^{2t} + 4\lambda te^{2t} - \left(\lambda e^{2t} + 2\lambda te^{2t}\right)$$

$$- 2\lambda te^{2t} = 3\lambda e^{2t}$$

Letting $\lambda = \frac{1}{3}$ gives a particular integral of

$v = \frac{1}{3}te^{2t}$.

Therefore the general solution is

$$v = Ae^{2t} + Be^{-t} + \frac{1}{3}te^{2t}$$

c $s = \frac{Ae^{2t} + Be^{-t}}{t} + \frac{1}{3}e^{2t}; t \neq 0$

4 a $\frac{dx}{dt} = u + t\frac{du}{dt}, \frac{d^2x}{dt^2} = 2\frac{du}{dt} + t\frac{d^2u}{dt^2}$

So differential equation becomes

$$t\left(2\frac{du}{dt} + t\frac{d^2u}{dt^2}\right) - 2\left(u + t\frac{du}{dt}\right) + \left(\frac{2 + t^2}{t}\right)ut = t^4$$

which rearranges to the required equation.

4 b The auxiliary equation is $m^2+1=0$

$\Rightarrow m=\pm i$

So the general solution is

$u=A\cos t+B\sin t+f(t)$

where $f(t)$ is a particular integral

Let $f(t)=at^2+bt+c$

$f'(t)=2at+b, f''(t)=2a$

Then the equation becomes

$2a+at^2+bt+c=t^2$

So $a=1, b=0, c=-2$

So the general solution is

$u=A\cos t+B\sin t+t^2-2$

$\Rightarrow x=t\left(A\cos t+B\sin t+t^2-2\right)$

c As t gets large, $\boldsymbol{x}$ gets large; the spring will reach its elastic limit and/or break.