Numerical Methods Mixed Exercise 8

1 $(x_0, y_0) = (2,1)$

$h = \frac{3-2}{2} = 0.5$

$\left(\frac{dy}{dx}\right)_0 = 1 \times e^{2\times2} - 2 \times \ln 1 = 54.59815\ldots$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.5 \times 54.59815$

$y_1 = 28.29908\ldots$

$(x_1, y_1) = (2.5, 29.29908\ldots)$

$\left(\frac{dy}{dx}\right)_1 = 29.29908\ldots \times e^{2\times2.5} - 2.5 \times \ln 29.29908\ldots = 4191.59805\ldots$

$y_2 = y_1 + h\left(\frac{dy}{dx}\right)_1$

$y_2 = 28.29908\ldots + 0.5 \times 4191.59805\ldots$

$y_2 = 2124.09810\ldots$

Therefore, at $f(3) = 2124.098$ (3 d.p.)

2 **a** $(x_0, y_0) = (0, 2)$

$y_1 = 2.6$

$\left(\frac{dy}{dx}\right)_0 = \left(0 + 2^2\right)^2 - \left(0^2 - 2\right)^2 = 12$

$\frac{y_1 - y_0}{h} = \left(\frac{dy}{dx}\right)_0$

$h = \frac{2.6 - 2}{12} = 0.05$

2 b $(x_1, y_1) = (0.05, 2.6)$

$\left(\frac{dy}{dx}\right)_1 = (0.05 + 2.6^2)^2 - (0.05^2 - 2.6)^2 = 39.62909\ldots$

$y_2 = y_1 + h\left(\frac{dy}{dx}\right)_1$

$y_2 = 2.6 + 0.05 \times 39.62909\ldots$

$y_2 = 4.58145\ldots = 4.581 \ (3 \text{ d.p.})$

$\left(\frac{dy}{dx}\right)_2 = \left(0.1 + (4.58145\ldots)^2\right)^2$

$-(0.1^2 - 4.58145\ldots)^2$

$= 423.87839\ldots$

$y_3 = y_2 + h\left(\frac{dy}{dx}\right)_2$

$y_3 = 4.58145\ldots + 0.05 \times 423.87839\ldots$

$y_3 = 25.77537\ldots = 25.775 \ (3 \text{ d.p.})$

3 $(t_0, x_0) = (2, 5)$

$h = \frac{3-2}{2} = 0.5$

$\left(\frac{dx}{dt}\right)_0 = \frac{5^2 - 2}{2 \times 5 - 2^2} = 3.83\ldots$

$\frac{x_1 - x_0}{h} = \left(\frac{dx}{dt}\right)_0$

$x_1 = 5 + 0.5 \times 3.83\ldots$

$x_1 = 6.92\ldots$

$(t_1, x_1) = (2.5, 6.92\ldots)$

$\left(\frac{dx}{dt}\right)_1 = \frac{(6.92\ldots)^2 - 2.5}{2.5 \times 6.92\ldots - 2.5^2} = 4.11\ldots$

$\frac{x_2 - x_1}{h} = \left(\frac{dx}{dt}\right)_1$

$x_2 = 6.92\ldots + 0.5 \times 4.11\ldots$

$x_2 = 8.97\ldots$

Therefore, the value of the commodity three days after it is purchased is £9000 (rounded to the nearest hundred pounds).

4 $(x_0, v_0) = (5, 8)$

$h = 0.25$

$$\left(\frac{dv}{dx}\right)_0 = \frac{2\times5-25.6}{3\times8} - 0.001\times8 = -0.658$$

$$v_1 = v_0 + h\left(\frac{dv}{dx}\right)_0$$

$v_1 = 8 + 0.25\times0.658$

$v_1 = 7.8355$

$(x_1, v_1) = (5.25, 7.8355)$

$$\left(\frac{dv}{dx}\right)_1 = \frac{2\times5.25-25.6}{3\times7.8355} - 0.001\times7.8355 = -0.650\ldots$$

$$v_2 = v_0 + 2h\left(\frac{dv}{dx}\right)_1$$

$v_2 = 8 + 2\times0.25\times-0.650\ldots$

$v_2 = 7.675\ldots$

$(x_2, v_2) = (5.5, 7.675\ldots)$

$$\left(\frac{dv}{dx}\right)_2 = \frac{2\times5.5-25.6}{3\times7.675\ldots} - 0.001\times7.675\ldots$$

$= -0.642\ldots$

$$v_3 = v_1 + 2h\left(\frac{dv}{dx}\right)_2$$

$v_3 = 7.8355 + 2\times0.25\times-0.642\ldots$

$v_3 = 7.515\ldots$

Therefore, at $x = 5.75$, $v \approx 7.52$ (2 d.p.)

5 a $(t_0, v_0) = (0, 2)$

$h = 0.1$

$$\left(\frac{dv}{dt}\right)_0 = 10\times0 - 2\times2 = -4$$

$$v_1 = v_0 + h\left(\frac{dv}{dt}\right)_0$$

$v_1 = 2 + 0.1\times-4$

$v_1 = 1.6$

b $(t_1, v_1) = (0.1, 1.6)$

$$\left(\frac{dv}{dt}\right)_1 = 10\times0.1 - 2\times1.6 = -2.2$$

$$v_2 = v_0 + 2h\left(\frac{dv}{dt}\right)_1$$

$v_2 = 2 + 2\times0.1\times-2.2$

$v_2 = 1.56$

5 c $\dfrac{\mathrm{d}v}{\mathrm{d}t}+2v=10t$

Integrating factor is $e^{\int 2dt}=e^{2t}$

Multiplying both sides by the integrating factor:

$$\left(\frac{\mathrm{d}v}{\mathrm{d}t}+2v\right)e^{2t}=10te^{2t}$$

$$\frac{d\left(e^{2t}v\right)}{dt}=10te^{2t}$$

$$e^{2t}v=\int 10te^{2t}\,\mathrm{d}t$$

$$e^{2t}v=5te^{2t}-\frac{5}{2}e^{2t}+c$$

Substituting $t=0$ and $v=2$ into the equation gives

$$2=0-\frac{5}{2}+c$$

$$c=\frac{9}{2}$$

Hence

$$v=5t-\frac{5}{2}+\frac{9}{2}e^{-2t}$$

d When $t=0.2$,

$$v=5\times 0.2-\frac{5}{2}+\frac{9}{2}e^{-2\times 0.2}$$

$$v=-\frac{3}{2}+\frac{9}{2}e^{-0.4}$$

Percentage error $=\dfrac{1.56-\left(-\dfrac{3}{2}+\dfrac{9}{2}e^{-0.4}\right)}{-\dfrac{3}{2}+\dfrac{9}{2}e^{-0.4}}\times 100=2.87\%$

6 $(x_0, y_0) = (1, 2)$

$h = 0.2$

$\left(\frac{dy}{dx}\right)_0 = 0.5$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 2 + 0.2 \times 0.5$

$y_1 = 2.1$

$(x_1, y_1) = (1.2, 2.1)$

$\left(\frac{d^2y}{dx^2}\right)_1 = 2\cos 1.2 - 2.1^3 + 3 = -5.536\ldots$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{d^2y}{dx^2}\right)_1$

$y_2 = 2 \times 2.1 - 2 + 0.2^2 \times -5.536\ldots$

$y_2 = 1.979\ldots$

$(x_2, y_2) = (1.4, 1.979\ldots)$

$\left(\frac{d^2y}{dx^2}\right)_2 = 2\cos 1.4 - (1.979\ldots)^3 + 3 = -4.405$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{d^2y}{dx^2}\right)_2$

$y_3 = 2 \times 1.979\ldots - 2.1 + 0.2^2 \times -4.405\ldots$

$y_3 = 1.681\ldots$

$y_3 \approx 1.681$ (3 d.p.)

7 a $(x_0, y_0) = (1, 3)$

$\left(\frac{dy}{dx}\right)_0 = 2$

$\left(\frac{d^2y}{dx^2}\right)_0 = \frac{1^3 - 3^2}{3 \times 1 \times 3} \times 2 = -\frac{16}{9} = -1.778\ldots$

7 b $h = 0.1$

$$\frac{y_1 - y_{-1}}{2h} = \left(\frac{dy}{dx}\right)_0$$

$$y_1 - y_{-1} = 2 \times 0.1 \times 2$$

$$y_1 - y_{-1} = 0.4$$

$$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{d^2y}{dx^2}\right)_0$$

$$y_1 + y_{-1} = 2 \times 3 + 0.1^2 \times -1.778\ldots$$

$$y_1 + y_{-1} = 5.982\ldots$$

Adding the two equations gives

$$2y_1 = 0.4 + 5.982\ldots = 6.382\ldots$$

$$y_1 = 3.191 \ (4 \text{ s.f.})$$

8 a $\int_{-2}^{-1} \left(\sin\left(x^2\right) + x^2\right) dx$

$$h = \frac{-1-(-2)}{4} = 0.25$$

x_i	y_i
−2	3.243…
−1.75	3.142…
−1.5	3.028…
−1.25	2.562…
−1	1.841…

$$\int_{-2}^{-1} \left(\sin\left(x^2\right) + x^2\right) dx$$

$$\approx \frac{1}{3} \times 0.25\left(3.243\ldots + 4\left(3.142\ldots + 2.562\ldots\right) + 2\left(3.028\ldots\right) + 1.841\ldots\right) = 2.830 \ (4 \text{ s.f.})$$

b The approximation could be improved using more intervals to estimate the integral.

9 a $\int_0^1 \frac{1}{1+\sin x}\,dx$

$h = \frac{1-0}{4} = 0.25$

x_i	y_i
0	1
0.25	0.802…
0.5	0.676…
0.75	0.595…
1	0.543…

$$\int_0^1 \frac{1}{1+\sin x}\,dx$$
$$\approx \frac{1}{3}\times 0.25\left(1+4\left(0.802\ldots+0.595\ldots\right)+2\left(0.676\ldots\right)+0.543\ldots\right) = 0.70668\ (5\text{ d.p.})$$

b By Weierstrass substitution, if $t = \tan\frac{x}{2}$

$$\int \frac{1}{1+\sin x}\,dx$$
$$= \int \frac{1}{1+\frac{2t}{1+t^2}}\frac{2}{1+t^2}\,dt$$
$$= \int \frac{2}{(1+t)^2}\,dt$$
$$= -\frac{2}{(1+t)}+c$$
$$= -\frac{2}{\left(1+\tan\frac{x}{2}\right)}+c \quad \int_0^1 \frac{1}{1+\sin x}\,dx = \left[-\frac{2}{\left(1+\tan\frac{x}{2}\right)}\right]_0^1$$
$$= 0.70659\ (5\text{ d.p.})$$

c Percentage error $= \frac{0.70668-0.70659}{0.70659}\times 100 = 0.013\%$ (2 s.f.)

Challenge

a Assume that the equation of the parabola is $y = ax^2 + bx + c$,
and also assume that $x_0 = -h$, $x_1 = 0$, $x_2 = h$
The area under the parabola is given by

$$\int_{-h}^{h} \left(ax^2 + bx + c\right) dx$$

$$= \left[\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right]_{-h}^{h}$$

$$= \frac{2ah^3}{3} + 2ch$$

Also for points on the parabola,

$$y_0 = ah^2 - bh + c$$

$$y_1 = c$$

$$y_2 = ah^2 + bh + c$$

Note that

$$y_0 + 4y_1 + y_2 = \left(ah^2 - bh + c\right) + 4c$$

$$+\left(ah^2 - bh + c\right) = 2ah^2 + 6c$$

Therefore, the area under the parabola is

$$\frac{2ah^3}{3} + 2ch = \frac{1}{3}h\left(y_0 + 4y_1 + y_2\right)$$

b If the interval $[x_0, x_n]$ is divided into an even number n of subintervals of equal length h

$$h = \frac{x_n - x_0}{n}$$

There are a total of $n+1$ points with the x-coordinates $x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh = x_n$,
and the corresponding y-coordinates are $y_0, y_1, y_2, \dots, y_n$
The area under the curve is given by adding the areas under the parabolic curves passing through each set of three successive points.
Area under the curve

$$\approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \dots$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \dots + 4y_{n-1} + y_n\right)$$

$$+ \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

$$= \frac{h}{3}\left(y_0 + 4\left(y_1 + y_3 + \dots\right) + 2\left(y_2 + y_4 + \dots\right) + y_n\right)$$