Numerical Methods 8D

1 $\int_2^4 \frac{\ln x}{x} dx$

$h = \frac{4-2}{4} = 0.5$

x_i	y_i
2	0.34657…
2.5	0.36652…
3	0.36620…
3.5	0.35793…
4	0.34657…

$\int_2^4 \frac{\ln x}{x} dx$

$\approx \frac{1}{3} \times 0.5\left(0.34657\ldots + 4\left(0.36652\ldots + 0.35793\ldots\right) + 2\left(0.36620\ldots\right) + 0.34657\ldots\right)$

$= 0.7206 \ (4 \text{ s.f.})$

2 $\int_0^3 \sqrt{1+x^5}\,dx$

$h = \frac{3-0}{6} = 0.5$

x_i	y_i
0	1
0.5	1.016…
1	1.414…
1.5	2.932…
2	5.744…
2.5	9.932…
3	15.620…

$\int_0^3 \sqrt{1+x^5}\,dx \approx \frac{1}{3} \times 0.5\left(1+4\left(1.016\ldots+2.932\ldots+9.932\ldots\right)+2\left(1.414\ldots+5.744\ldots\right)+15.620\ldots\right)$

$= 14.41 \ (4 \text{ s.f.})$

3 $\int_0^1 \sqrt{\cos x + \tan x}\,dx$

$h = \frac{1-0}{4} = 0.25$

x_i	y_i
0	1
0.25	1.106…
0.5	1.193…
0.75	1.290…
1	1.448…

$$\int_0^1 \sqrt{\cos x + \tan x}\,dx \approx \frac{1}{3} \times 0.25 \begin{pmatrix} 1 + 4(1.016\ldots + 1.290\ldots) \\ +2(1.193\ldots) + 1.448\ldots \end{pmatrix} = 1.202 \ (4 \text{ s.f.})$$

b Increasing the number of intervals would give a better estimate for the approximation.

4 a Simpson's rule is only valid for even number of intervals since the intervals are paired off in order to use a quadratic to approximate the curve.

4 b $\int_1^2 1 - \ln\left(1+\cos^2 x\right) dx$

$h = \dfrac{2-1}{8} = 0.125$

x_i	y_i
1	0.74386…
1.125	0.82949…
1.250	0.90521…
1.375	0.96245…
1.500	0.99501…
1.625	0.99707…
1.750	0.96872…
1.875	0.91408…
2	0.84028…

$$\approx \frac{1}{3} \times 0.125 \begin{pmatrix} 0.74386\ldots + 4\left(0.82949\ldots + 0.96245\ldots + 0.99707\ldots + 0.91408\ldots\right) \\ +2\left(0.90521\ldots + 0.99501\ldots + 0.96872\ldots\right) + 0.84028\ldots \end{pmatrix}$$

$= 0.9223 \quad (4 \text{ d.p.})$

5 a $\int_{0.5}^{1.5} \frac{1}{\sin x + \tan x}\, dx$

$h = 0.25$

x_i	y_i
0.5	0.97492…
0.75	0.61987…
1	0.41686…
1.25	0.25262…
1.5	0.06623…

$$\approx \frac{1}{3} \times 0.25\left(0.97492\ldots + 4\left(0.61987\ldots + 0.25262\ldots\right) + 2\left(0.41686\ldots\right) + 0.06623\ldots\right) = 0.4471 \ (4 \text{ s.f.})$$

b By Weierstrass substitution, if $t = \tan\frac{x}{2}$

$$\int \frac{1}{\sin x + \tan x}\, dx$$

$$= \int \frac{1}{\frac{2t}{1+t^2} + \frac{2t}{1-t^2}} \frac{2}{1+t^2}\, dt$$

$$= \int \frac{\left(1+t^2\right)\left(1-t^2\right)}{2t\left(1-t^2+1+t^2\right)} \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1-t^2}{2t}\, dt = \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + c$$

$$\int_{0.5}^{1.5} \frac{1}{\sin x + \tan x}\, dx$$

$$= \left[\frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2}\right]_{0.5}^{1.5}$$

$$= 0.44647999...$$

$$\approx 0.44648 \ (5 \text{ d.p.})$$

c Percentage error $= \frac{0.4471 - 0.4465\ldots}{0.4465\ldots} \times 100 = 0.14\% \ (2 \text{ d.p.})$

6 a $\int_1^3 x\sinh x\,dx$

$h = \frac{3-1}{4} = 0.5$

x_i	y_i
1	1.1752…
1.5	3.1939…
2	7.2537…
2.5	15.1255…
3	30.0536…

$$\int_1^3 x\sinh x\,dx \approx \frac{1}{3}\times 0.5(1.1752\ldots + 4(3.1939\ldots + 15.1255\ldots) + 2(7.2537\ldots) + 30.0536\ldots) = 19.84 \ (4 \text{ s.f.})$$

b $\int_1^3 x\sinh x\,dx$

$= [x\cosh x]_1^3 - \int_1^3 \cosh x\,dx$

$= [x\cosh x - \sinh x]_1^3$

$$= = \left(\frac{3e^3 + 3e^{-3}}{2} - \frac{e^3 - e^{-3}}{2}\right) - \left(\frac{e^1 + e^{-1}}{2} - \frac{e^1 - e^{-1}}{2}\right) = e^3 + 2e^{-3} - e^{-1}$$

c $\int_1^3 x\sinh x\,dx$

$= e^3 + 2e^{-3} - e^{-1}$

$= 19.817...$

Percentage error $= \frac{19.84 - 19.817\ldots}{19.817\ldots}\times 100 = 0.115\%$

7 a $x = t + t^2$, $y = t - t^2$

$x = 0 \Rightarrow t = 0$, $x = 2 \Rightarrow t = 1$

$dx = (1 + 2t)\,dt$

$$\text{Volume of solid} = \pi\int_0^2 y^2\,dx$$

$$= \pi\int_0^1 (t - t^2)^2 (1 + 2t)\,dt$$

$$= \pi\int_0^1 (1 + 2t)(t^2 - 2t^3 + t^4)\,dt$$

$$= \pi\int_0^1 1(t^2 - 2t^3 + t^4) + 2t(t^2 - 2t^3 + t^4)\,dt$$

$$= \pi\int_0^1 (t^2 - 2t^3 + t^4 + 2t^3 - 4t^4 + 2t^5)\,dt$$

$$= \pi\int_0^1 (2t^5 - 3t^4 + t^2)\,dt$$

$$= \pi\int_0^1 t^2(2t^3 - 3t^2 + 1)\,dt$$

b $h = \dfrac{1-0}{2} = 0.5$

x_i	y_i
0	0
0.25	1.166…
0.5	0.393…
0.75	0.276…
1	0

$$\pi\int_0^1 (2t^5 - 3t^4 + t^2)\,dt$$

$$\approx \frac{1}{3} \times 0.25\begin{pmatrix} 0 + 4\times(0.166 + 0.276) \\ +2\times 0.393 + 0 \end{pmatrix}$$

$$= 0.2127 \ (4 \text{ s.f.})$$

7 c Exact area $= \pi \int_0^1 \left(2t^5 - 3t^4 + t^2\right) dt$

$$= \pi \left[\frac{t^6}{3} - \frac{3t^5}{5} + \frac{t^3}{3} \right]_0^1 = \frac{\pi}{15}$$

Percentage error $= \dfrac{0.2127\ldots - \dfrac{\pi}{15}}{\dfrac{\pi}{15}} \times 100 = 1.56\ldots\% < 1.6\%$

d The approximation can be improved by using more intervals to estimate the integral.