Methods in Calculus Mixed Exercise 7

1 a $y=(3x^2+2x)(x^3+2x-6)$

$u=3x^2+2x, \frac{du}{dx}=6x+2, \frac{d^2u}{dx^2}=6$

$v=x^3+2x-6, \frac{dv}{dx}=3x^2+2, \frac{d^2v}{dx^2}=6x$

$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$

$=6x(3x^2-2x)+2(6x-2)(3x^2+2)$

$+6(x^3+2x-6)$

$=60x^3-24x^2+36x-44$

b $y=e^{4x}\tan 2x$

$u=e^{4x}, \frac{du}{dx}=4e^{4x}, \frac{d^2u}{dx^2}=16e^{4x}$

$v=\tan 2x, \frac{dv}{dx}=2\sec^2 2x, \frac{d^2v}{dx^2}$

$=8\tan 2x\sec^2 2x$

$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$

$=8e^{4x}(\tan 2x\sec^2 2x+2\tan 2x+2\sec^2 2x)$

c $y=x^{\frac{3}{2}}\arctan 2x$

$u=x^{\frac{3}{2}}, \frac{du}{dx}=\frac{3}{2}x^{\frac{1}{2}}, \frac{d^2u}{dx^2}=\frac{3}{4}x^{-\frac{1}{2}}$

$\frac{d^3u}{dx^3}=-\frac{3}{8}x^{-\frac{3}{2}}$

$v=\arctan 2x, \frac{dv}{dx}=\frac{2}{1+4x^2}, \frac{d^2v}{dx^2}$

$=\frac{-2}{\left(1+4x^2\right)^2}8x$

$\frac{d^3v}{dx^3}=\frac{-16}{\left(1+4x^2\right)^2}+\frac{32x}{\left(1+4x^2\right)^3}8x$

$\frac{d^3y}{dx^3}=u\frac{d^3v}{dx^3}+3\frac{d^2u}{dx^2}\frac{dv}{dx}+3\frac{du}{dx}\frac{d^2v}{dx^2}+v\frac{d^3u}{dx^3}$

$=\frac{256x^2}{\left(1+4x^2\right)^3}x^{\frac{3}{2}}-\frac{16}{\left(1+4x^2\right)^3}x^{\frac{3}{2}}-\frac{72x}{\left(1+4x^2\right)^3}x^{\frac{1}{2}}$

$+\frac{9x^{\frac{1}{2}}}{1+4x^2}-\frac{3}{8x\sqrt{x}}\arctan 2x$

$=\frac{-96x^{\frac{7}{2}}-88x^{\frac{3}{2}}}{\left(1+4x^2\right)^3}+\frac{9x^{-\frac{1}{2}}}{1+4x^2}-\frac{3}{8x\sqrt{x}}\arctan 2x$

2 $y=\tan x=\frac{\sin x}{\cos x}$

$u=\sin x, \frac{du}{dx}=\cos x, \frac{d^2u}{dx^2}=-\sin x$

$v=(\cos x)^{-1}, \frac{dv}{dx}=\frac{\sin x}{\cos x}$

$\frac{d^2v}{dx^2}=\frac{\cos x}{(\cos x)^2}-\frac{2\sin x}{(\cos x)^3}(-\sin x)$

$=\frac{1}{\cos x}\left(1+2\frac{\sin^2 x}{\cos^2 x}\right)$

$\frac{d^2y}{dx^2}=u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$

$=2\sec^2(x)\tan(x)$

3 a By Leibnitz's theorem

$(\mathrm{f}(\mathrm{gh}))''(x)$

$= \mathrm{f}''(x)(\mathrm{gh})(x) + 2\mathrm{f}'(x)(\mathrm{gh})'(x) + \mathrm{f}(x)(\mathrm{gh})''(x)$

$= \mathrm{f}''(x)\mathrm{g}(x)\mathrm{h}(x) + 2\mathrm{f}'(x)(\mathrm{g}'(x)\mathrm{h}(x) + \mathrm{g}(x)\mathrm{h}'(x)) + \mathrm{f}(x)(\mathrm{g}''(x)\mathrm{h}(x) + 2\mathrm{g}'(x)\mathrm{h}'(x) + \mathrm{g}(x)\mathrm{h}''(x))$

b $y = e^x \sin 2x \cos 3x$

$f = e^x = f' = f''$

$g = \sin 2x, g' = 2\cos 2x, g'' = -4\sin 2x$

$h = \cos 3x, h' = -3\sin 3x, h'' = -9\cos 3x$

$y'' = e^x[-9\sin 2x\cos 3x - 4\sin 2x\cos 3x$
$+\sin 2x\cos 3x + 2(2\cos 2x\cos 3x$
$-6\cos 2x\sin 3x - 3\sin 2x\sin 3x)]$
$= 2e^x[-6\sin 2x\cos 3x + 2\cos 2x\cos 3x$
$-6\cos 2x\sin 3x - 3\sin 2x\sin 3x]$

4 $y = \dfrac{\sqrt{3x+2}}{\cos x}$

$u = (3x+2)^{\frac{1}{2}}, \dfrac{du}{dx} = \dfrac{3}{2}(3x+2)^{-\frac{1}{2}}, \dfrac{d^2u}{dx^2}$
$= -\dfrac{9}{4}(3x+2)^{-\frac{3}{2}}$

$\dfrac{d^3u}{dx^3} = \dfrac{81}{8}(3x+2)^{-\frac{5}{2}}$

$v = \sec x, \dfrac{dv}{dx} = \tan x\sec x$

$\dfrac{d^2v}{dx^2} = \sec^3 x + \sec x\tan^2 x$

$\dfrac{d^3v}{dx^3} = 5\sec^3 x\tan x + \tan^3 x\sec x$

$\dfrac{d^3y}{dx^3} = u\dfrac{d^3v}{dx^3} + 3\dfrac{d^2u}{dx^2}\dfrac{dv}{dx} + 3\dfrac{du}{dx}\dfrac{d^2v}{dx^2} + v\dfrac{d^3u}{dx^3}$
$= (3x+2)^{\frac{1}{2}}(5\sec^3 x\tan x + \tan^3 x\sec x)$
$+\dfrac{9}{2}(3x+2)^{-\frac{1}{2}}(\sec^3 x + \sec x\tan^2 x)$
$-\dfrac{27}{4}(3x+2)^{-\frac{3}{2}}\tan x\sec x + \dfrac{81}{8}(3x+2)^{-\frac{5}{2}}\sec x$
$= \sqrt{3x+2}(5\sec^3 x\tan x + \tan^3 x\sec x)$
$+\dfrac{9\sec^3 x + \sec x\tan^2 x}{2\sqrt{3x+2}} - \dfrac{27\tan x\sec x}{4(3x+2)^{\frac{3}{2}}} + \dfrac{81\sec x}{8(3x+2)^{\frac{5}{2}}}$

5 a $y = \sin x$

$\dfrac{dy}{dx} = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$

Assume $\dfrac{d^k y}{dx^k} = \sin\left(x + \dfrac{k\pi}{2}\right)$

Then $\dfrac{d^{k+1}y}{dx^{k+1}} = \cos\left(x + \dfrac{k\pi}{2}\right)$
$= \sin\left(x + \dfrac{(k+1)\pi}{2}\right)$

True for $n = k+1$ if true for $n = k$

True for $n = 1$ so true for $\forall n \in \mathbb{N}$

b $y = x^2 \sin x$

$\dfrac{d^n y}{dx^n} = \sum_{k=0}^{n}\binom{n}{k}\dfrac{d^k u}{dx^k}\dfrac{d^{n-k}v}{dx^{n-k}}$

$u = x^2, \dfrac{du}{dx} = 2x, \dfrac{d^2u}{dx^2} = 2$

$\dfrac{d^k u}{dx^k} = 0, k > 2$

$v = \sin x, \dfrac{d^m v}{dx^m} = \sin\left(x + \dfrac{m\pi}{2}\right)$

$\dfrac{d^n y}{dx^n} = x^2\sin\left(x + \dfrac{n\pi}{2}\right) + 2nx\sin\left(x + \dfrac{(n-1)\pi}{2}\right)$
$+2\binom{n}{2}\sin\left(x + \dfrac{(n-2)\pi}{2}\right)$
$= (x^2 - n(n-1))\sin\left(x + \dfrac{n\pi}{2}\right) - 2nx\cos\left(x + \dfrac{n\pi}{2}\right)$

6 a $f(x) = e^{2x} - 1, \quad f(0) = 0, \quad f'(x) = 2e^{2x}$

$g(x) = 3x, \quad g(0) = 0, \quad g'(x) = 3$

$\lim_{x\to 0}\dfrac{e^{2x}-1}{3x} = \lim_{x\to 0}\dfrac{2e^{2x}}{3} = \dfrac{2}{3}$

b $f(x) = \tan x, \quad f(0) = 0, \quad f'(x) = \sec^2 x$

$g(x) = 2x + \sin x, \quad g(0) = 0, \quad g'(x) = 2 + \cos x$

$\lim_{x\to 0}\dfrac{\tan x}{2x + \sin x} = \lim_{x\to 0}\dfrac{\sec^2 x}{2 + \cos x} = \dfrac{1}{3}$

6 c $f(x)=x^2+x-2,\ f(1)=0,\ f'(x)=2x+1$

$g(x)=x\ln x,\quad g(1)=0,\quad g'(x)=\ln x+1$

$$\lim_{x\to1}\frac{x^2+x-2}{x\ln x}=\lim_{x\to1}\frac{2x+1}{\ln x+1}=3$$

d $f(x)=\sin\pi x,\ f(1)=0,\ f'(x)=\pi\cos\pi x$

$g(x)=x^2+7x-8,\ g(1)=0,\ g'(x)$

$=2x+7$

$$\lim_{x\to1}\frac{\sin\pi x}{x^2+7x-8}=\lim_{x\to1}\frac{\pi\cos\pi x}{2x+7}=-\frac{\pi}{9}$$

7 $f(x)=\cos\frac{1}{2}x, f(\pi)=0, f'(x)=-\frac{1}{2}\sin\frac{1}{2}x$

$g(x)=x-\pi, g(\pi)=0, g'(x)=1$

$$\lim_{x\to\pi}\frac{\cos\frac{1}{2}x}{x-\pi}=\lim_{x\to\pi}\frac{-\frac{1}{2}\sin\frac{1}{2}x}{1}=-\frac{1}{2}$$

8 $f(x)=x-2, f(2)=0, f'(x)=1$

$g(x)=x^n-2^n, g(2)=0, g'(x)=nx^{n-1}$

$$\lim_{x\to2}\frac{x-2}{x^n-2^n}=\lim_{x\to2}\frac{1}{nx^{n-1}}=\frac{1}{n2^{n-1}}$$

9 $\lim\limits_{x\to\infty}(1+ax)^{\frac{1}{x}}=e^{\lim\limits_{x\to\infty}\frac{1}{x}\ln(1+ax)}$

$\lim\limits_{x\to\infty}\frac{1}{x}\ln(1+ax)=0$

as $\lim\limits_{x\to\infty}x=\infty, \lim\limits_{x\to\infty}\ln(1+ax)=\infty$

$f(x)=\ln(1+ax)$,

$f(\infty)=\lim\limits_{x\to\infty}\ln(1+ax)=\infty, f'(x)=\frac{a}{1+ax}$

$g(x)=x, g(\infty)=\lim\limits_{x\to\infty}x=\infty, g'(x)=1$

$$\lim_{x\to\infty}\frac{\ln(1+ax)}{x}=\lim_{x\to\infty}\frac{a}{(1+ax)}=0$$

$\lim\limits_{x\to\infty}\frac{1}{x}\ln(1+ax)=0$

$\lim\limits_{x\to\infty}(1+ax)^{\frac{1}{x}}=e^0=1$

10 a $t=\tan\frac{x}{2},\quad dx=\frac{2}{1+t^2}dt$

$$\int\frac{3}{2+4\cos x}dx=\int\frac{3}{2+4\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt$$

$$=\int\frac{6}{2(1+t^2)+4(1-t^2)}dt=\int\frac{3}{3-t^2}dt$$

$$=\frac{\sqrt{3}}{2}\int\frac{1}{\sqrt{3}-t}+\frac{1}{\sqrt{3}+t}dt$$

$$=\frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|+c$$

$$=\frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|+c$$

b $$\int\frac{\sec x}{\sin x+2\cos x}dx=\int\frac{\frac{1+t^2}{1-t^2}}{\frac{2t}{1+t^2}+2\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt$$

$$=\int\frac{1+t^2}{(t+1-t^2)(1-t^2)}dt=I$$

$$\frac{1+t^2}{(t+1-t^2)(1-t^2)}=\frac{A+Bt}{t+1-t^2}+\frac{C+Dt}{1-t^2}$$

$A=1, B=-2, C=0, D=2$

$$I=\int\frac{1-2t}{t+1-t^2}+\frac{2t}{1-t^2}dt$$

$$=\ln|t+1-t^2|-\ln|1-t^2|+c$$

$$=\ln\left|\frac{1+\tan\frac{x}{2}-\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right|+c$$

$$=\ln\left|\frac{\tan^2\frac{x}{2}-\tan\frac{x}{2}-1}{\tan^2\frac{x}{2}-1}\right|+c$$

c $$\int\frac{2}{\sin x+\cos x}dx=\int\frac{2}{\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt$$

$$=\int\frac{4}{1-t^2+2t}dt$$

$$=\int\frac{2}{\sqrt{2}}\left(\frac{1}{t-1+\sqrt{2}}-\frac{1}{t-1-\sqrt{2}}\right)dt$$

$$=\sqrt{2}\ln\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right|+c$$

$$=\sqrt{2}\ln\left|\frac{\tan\frac{x}{2}-1+\sqrt{2}}{\tan\frac{x}{2}-1-\sqrt{2}}\right|+c$$

11 a

$$\int_0^{\frac{\pi}{2}} \frac{2}{7+2\sin x+8\cos x}\,dx = \int_0^1 \frac{2}{7+2\frac{2t}{1+t^2}+8\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{4}{15+4t-t^2}\,dt = 4\int_0^1 \frac{1}{(\sqrt{19}+2-t)(\sqrt{19}-2+t)}\,dt$$

$$= 4\int_0^1 \frac{\frac{1}{2\sqrt{19}}}{(\sqrt{19}+2-t)} + \frac{\frac{1}{2\sqrt{19}}}{(\sqrt{19}-2+t)}\,dt$$

$$= \frac{2}{\sqrt{19}}\Big[-\ln\left|\sqrt{19}+2-t\right| + \ln\left|\sqrt{19}-2+t\right|\Big]_0^1$$

$$= \frac{2}{\sqrt{19}}\left[\ln\left|\frac{\sqrt{19}-2+t}{\sqrt{19}+2-t}\right|\right]_0^1 = \frac{2}{\sqrt{19}}\ln\left(\frac{\left(\sqrt{19}-1\right)\left(\sqrt{19}+2\right)}{\left(\sqrt{19}+1\right)\left(\sqrt{19}-2\right)}\right)$$

$$= \frac{2}{\sqrt{19}}\ln\left(\frac{154+17\sqrt{19}}{135}\right) = 0.2407 \text{ (4 d.p.)}$$

b $$\int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2-2\cos x}\,dx = \int_{\sqrt{3}}^{\infty} \frac{1}{2-2\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt$$

$$\int_{\sqrt{3}}^{\infty} \frac{1}{2t^2}\,dt = -\left[\frac{1}{2t}\right]_{\sqrt{3}}^{\infty} = \frac{1}{2\sqrt{3}}$$

$$= 0.2887 \text{ (4 d.p.)}$$

12 a $$\int \frac{1}{4\cos x-3\sin x}\,dx = \int \frac{1}{4\frac{1-t^2}{1+t^2}-3\frac{2t}{1+t^2}}\frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{4(1-t^2)-6t}\,dt = \int \frac{1}{2-2t^2-3t}\,dt$$

$$= \int \frac{-1}{2t^2+3t-2}\,dt$$

b $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{4\cos x-3\sin x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{2-2t^2-3t}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{5}\left(\frac{1}{t+2}-\frac{-2}{1-2t}\right)dt$$

$$= \frac{1}{5}\left[\ln\left|\frac{t+2}{1-2t}\right|\right]_{\frac{1}{\sqrt{3}}}^{1} = \frac{1}{5}\left(\ln 3 - \ln\left(\frac{\frac{1}{\sqrt{3}}+2}{1-\frac{2}{\sqrt{3}}}\right)\right)$$

$$= \frac{1}{5}\ln\left(\frac{6-3\sqrt{3}}{1+2\sqrt{3}}\right) = -0.3429 \text{ (4 d.p.)}$$

13 $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1-\operatorname{cosec} x}{\sin x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1-\frac{1+t^2}{2t}}{\frac{2t}{1+t^2}}\frac{2}{1+t^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1-\frac{1}{2t}-\frac{t}{2}}{t}\,dt = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{t}-\frac{1}{2t^2}-\frac{1}{2}\,dt$$

$$= \left[\ln t+\frac{1}{2t}-\frac{t}{2}\right]_{\frac{1}{\sqrt{3}}}^{1} = -\ln\frac{1}{\sqrt{3}}-\frac{\sqrt{3}}{2}+\frac{1}{2\sqrt{3}}$$

$$= \ln\sqrt{3}-\frac{1}{\sqrt{3}}$$

Challenge

$n=1$, $\lim\limits_{x\to\infty}\frac{x}{e^x} = \lim\limits_{x\to\infty}\frac{1}{e^x} = 0$ (as $\lim\limits_{x\to\infty} x = \lim\limits_{x\to\infty} \mathrm{e}^x = \infty$)

Assume true for $n=k$: $\lim\limits_{x\to\infty}\frac{x^k}{e^x} = 0$

Then $\lim\limits_{x\to\infty}\frac{x^{k+1}}{e^x} = \lim\limits_{x\to\infty}\frac{(k+1)x^k}{e^x}$

$= (k+1)\lim\limits_{x\to\infty}\frac{x^k}{e^x} = 0$

(by L'Hospital's rule, as $\lim\limits_{x\to\infty} x^{k+1} = \lim\limits_{x\to\infty}\mathrm{e}^x = \infty$)

True for $n=k+1$ if true for $n=k$

True for $n=1$ so true for $\forall n \in \mathbb{N}$