Methods in Calculus 7C

1 a $t=\tan\frac{x}{2},\quad dx=\frac{2}{1+t^2}dt$

$$\int\frac{1}{1+3\cos x}dx=\int\frac{1}{1+3\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt$$
$$=\int\frac{2}{1+t^2+3(1-t^2)}dt=\int\frac{1}{2-t^2}dt$$
$$=\frac{1}{2\sqrt{2}}\int\frac{1}{\sqrt{2}-t}+\frac{1}{\sqrt{2}+t}dt$$
$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|+c$$
$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\tan\frac{x}{2}}{\sqrt{2}-\tan\frac{x}{2}}\right|+c$$

b $$\int\sec x\,dx=\int\frac{1+t^2}{1-t^2}\frac{2}{1+t^2}dt$$
$$=\int\frac{2}{1-t^2}dt=\int\frac{1}{1+t}+\frac{1}{1-t}dt$$
$$=\ln\left|\frac{1+t}{1-t}\right|+c=\ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|+c$$

c $$\int\frac{1}{\sin x+\tan x}dx=\int\frac{1}{\frac{2t}{1+t^2}+\frac{2t}{1-t^2}}\frac{2}{1+t^2}dt$$
$$=\int\frac{1}{t}\frac{1-t^2}{1-t^2+1+t^2}dt=\int\frac{1}{2t}-\frac{t}{2}dt$$
$$=\frac{1}{2}\ln|t|-\frac{1}{4}t^2+c=\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|-\frac{1}{4}\tan^2\frac{x}{2}+c$$

d $$\int\frac{2}{1-\sin x}dx=\int\frac{2}{1-\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt$$
$$=\int\frac{4}{1+t^2-2t}dt=\int\frac{4}{(1-t)^2}dt$$
$$=\frac{4}{1-t}+c=\frac{4}{1-\tan\frac{x}{2}}+c$$

2 a $$\int_0^{\frac{\pi}{2}}\frac{\sec x}{1+\tan x}dx=\int_0^1\frac{\frac{1+t^2}{1-t^2}}{1+\frac{2t}{1-t^2}}\frac{2}{1+t^2}dt$$
$$=\int_0^1\frac{2}{1-t^2+2t}dt$$
$$=\int_0^1\frac{1}{\sqrt{2}}\left(\frac{1}{t-1+\sqrt{2}}-\frac{1}{t-1-\sqrt{2}}\right)dt$$
$$=\left[\frac{1}{\sqrt{2}}\ln\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right|\right]_0^1$$
$$=-\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)=1.2465\text{ (4 d.p.)}$$

b $$\int_0^{\frac{\pi}{2}}\frac{1-\cos x}{1+\sin x+2\cos x}dx$$
$$=\int_0^1\frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+2\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt$$
$$=\int_0^1\frac{4t^2}{(3+2t-t^2)(1+t^2)}dt=I$$
$$\frac{4t^2}{(3+2t-t^2)(1+t^2)}=\frac{A}{t-3}+\frac{B}{t+1}+\frac{C+Dt}{1+t^2}$$
$$A=-\frac{9}{10},B=\frac{1}{2},C=-\frac{4}{5},D=\frac{2}{5}$$
$$I=\int_0^1\frac{-\frac{9}{10}}{t-3}+\frac{\frac{1}{2}}{t+1}+\frac{-\frac{4}{5}}{1+t^2}+\frac{\frac{2}{5}t}{1+t^2}dt$$
$$=\left[-\frac{9}{10}\ln|t-3|+\frac{1}{2}\ln|t+1|-\frac{4}{5}\arctan x+\frac{1}{5}\ln(1+t^2)\right]_0^1$$
$$=\frac{9}{10}\ln3-\frac{1}{5}\ln2-\frac{\pi}{5}=0.2218\text{ (4 d.p.)}$$

c $$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx=\int_{\frac{1}{\sqrt{3}}}^1\frac{\frac{2t}{1+t^2}}{1+\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}dt$$
$$=\int_{\frac{1}{\sqrt{3}}}^1\frac{4t}{\left(1+t^2\right)^2+\left(1-t^2\right)^2}dt=\int_{\frac{1}{\sqrt{3}}}^1\frac{2t}{1+t^4}dt$$
$$=\int_{\frac{1}{3}}^1\frac{1}{1+u^2}du,\quad u=t^2,du=2tdt$$
$$=\left[\arctan u\right]_{\frac{1}{3}}^1=\frac{\pi}{4}-\arctan\frac{1}{3}=0.4636\text{ (4 d.p.)}$$

2 d $\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{\cot x}{1+\operatorname{cosec} x} dx = \int_1^{\sqrt{3}} \frac{\frac{1-t^2}{2t}}{1+\frac{1+t^2}{2t}} \frac{2}{1+t^2} dt = I$

$= \int_1^{\sqrt{3}} \frac{1-t^2}{2t+1+t^2} \frac{2}{1+t^2} dt = \int_1^{\sqrt{3}} \frac{1-t}{1+t} \frac{2}{1+t^2} dt$

$\frac{2(1-t)}{(1+t)(1+t^2)} = \frac{A}{t+1} + \frac{B+Ct}{1+t^2}$

$A = 2, B = 0, C = -2$

$I = \int_1^{\sqrt{3}} \frac{2}{t+1} - \frac{2t}{1+t^2} dt$

$= \left[2\ln|t+1| - \ln(1+t^2)\right]_1^{\sqrt{3}}$

$= 2\ln(1+\sqrt{3}) - 3\ln 2 = -0.0693$ (4 d.p.)

3 a $\int \frac{1}{12-13\sin x} dx = \int \frac{1}{12-13\frac{2t}{1+t^2}} \frac{2}{1+t^2} dt$

$= \int \frac{2}{12+12t^2-26t} dt = \int \frac{1}{6+6t^2-13t} dt$

b $\int_0^{\frac{\pi}{3}} \frac{1}{12-13\sin x} dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{6-13t+6t^2} dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{A}{3-2t} + \frac{B}{2-3t} dt$

$\frac{1}{6-13t+6t^2} = \frac{A}{3-2t} + \frac{B}{2-3t}$

$A = -\frac{2}{5}, B = \frac{3}{5}$

$\int_0^{\frac{1}{\sqrt{3}}} \frac{-\frac{2}{5}}{3-2t} + \frac{\frac{3}{5}}{2-3t} dt = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{5}\frac{(-2)}{3-2t} - \frac{1}{5}\frac{(-3)}{2-3t} dt$

$= \left[\frac{1}{5}\log(3-2t) - \frac{1}{5}\log(2-3t)\right]_0^{\frac{1}{\sqrt{3}}} = \left[\frac{1}{5}\log\left(\frac{3-2t}{2-3t}\right)\right]_0^{\frac{1}{\sqrt{3}}}$

$= \frac{1}{5}\log\left(\frac{3-\frac{2}{\sqrt{3}}}{2-\sqrt{3}}\right) - \frac{1}{5}\log\left(\frac{3}{2}\right) = \frac{1}{5}\log\left(\frac{2}{9}(12+5\sqrt{3})\right)$

$= 0.3048$ (4 d.p.)

4 $I = \int_0^{\frac{\pi}{2}} \frac{1}{a+\cos x} dx = \frac{\pi}{3\sqrt{3}}$

$= \int_0^1 \frac{1}{a+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int_0^1 \frac{2}{a+1+(a-1)t^2} dt$

$= \int_0^1 \frac{2}{\sqrt{a+1}\sqrt{a-1}} \frac{\frac{\sqrt{a+1}}{\sqrt{a-1}}}{\frac{a+1}{a-1}+t^2} dt$ (assume $a > 1$)

$= \frac{2}{\sqrt{a+1}\sqrt{a-1}} \left[\arctan \frac{t}{\frac{\sqrt{a+1}}{\sqrt{a-1}}}\right]_0^1$

$= \frac{2}{\sqrt{a+1}\sqrt{a-1}} \arctan \frac{\sqrt{a-1}}{\sqrt{a+1}}$

$a = 2 \Rightarrow I = \frac{2}{\sqrt{3}} \arctan \frac{1}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$

5 $I = \int_0^1 \frac{\arccos \frac{1-x^2}{1+x^2}}{1+x^2} dx$

$t = \arccos \frac{1-x^2}{1+x^2}, \cos t = \frac{1-x^2}{1+x^2} = -1 + \frac{2}{1+x^2}$

$\sin t\ dt = \frac{2\times 2x}{(1+x^2)^2} dx, \sin t = \frac{2x}{1+x^2}$

$dt = \frac{2}{1+x^2} dx, \frac{1}{2} dt = \frac{1}{1+x^2} dx$

$I = \int_0^{\frac{\pi}{2}} t \times \frac{1}{2} dt = \left[\frac{t^2}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{16}$

Challenge

$I = \int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx$

$x = \sin\theta, dx = \cos\theta d\theta, d\theta = \frac{1}{\sqrt{1-x^2}} dx$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1-\sin^2\theta}{1+\sin^2\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sec^2\theta + \tan^2\theta} d\theta$

$t = \tan\theta, dt = \sec^2\theta d\theta, d\theta = \frac{1}{1+t^2} dt$

$I = \int_{-\infty}^{\infty} \frac{1}{1+2t^2} \frac{1}{1+t^2} dt = \int_{-\infty}^{\infty} \frac{2}{1+2t^2} - \frac{1}{1+t^2} dt$

$= \left[\sqrt{2}\arctan\sqrt{2}t - \arctan t\right]_{-\infty}^{\infty} = 2(\sqrt{2}\frac{\pi}{2} - \frac{\pi}{2})$

$= (\sqrt{2}-1)\pi$